# Basic Course Math

Version 1.1
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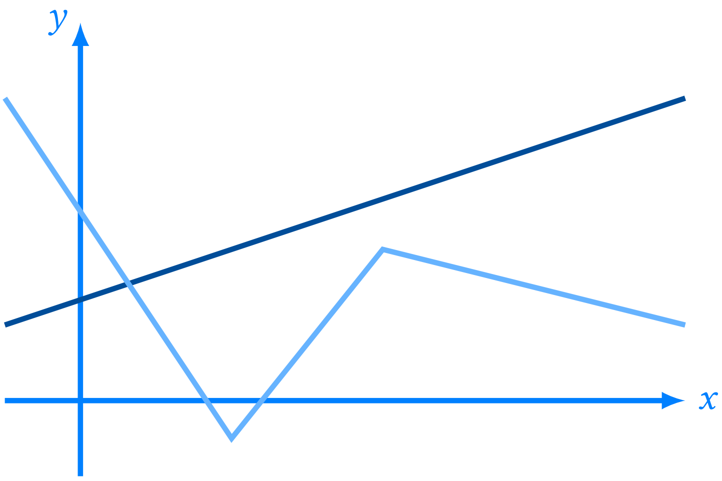



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Version 1.1, 2019

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The document is written using the document preparation system LaTeX, see www.tug.org and www.miktex.org. Figures and diagrams are produced using *pgf/TikZ*, see www.ctan.org/pkg/pgf.

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# Contents

# 1 Variables and relations

A lot of mathematics deals to some extent with relations between different quantities or objects. E.g., if we look at a circle, the area and the radius are related. If we take a taxi, there's a relationship between the length of the fare and its price. Relationships such as these may be expressed mathematically using formulas and equations.

When we use a formula to describe a relationship, we use *variables* to describe the different, related quantities.

## 1.1 Variables

A variable is—as the name implies—a quantity which varies.[1] So, it is a quantity with no fixed value. A variabled in mathematics is always denoted by *one* letter,[2] e.g. $x$.

We may determine the area of a circle using the formula

$$A = \pi \cdot r^2 \,, \tag{1.1}$$

where $A$ denotes the area, and $r$ the radius. In this example, both $A$ and $r$ are variables, i.e. $A$ and $r$ have no fixed value.[3] But a relationship exists between the values of the two variables—if we look at a circle with a certain radius, we also have a certain area. This relationship is what the formula describes.

When two variables are related, changing the value of one of the variables will lead to a change in the other variable. E.g. if we increase the radius of a circle, we also increase the area. In the formula (1.1), we call $r$ the *independent variable* while $A$ is called the *dependent variable*, because its value depends on the value of $r$.

In principle, we might choose to call $A$ the independent variable, and $r$ the dependent—because if we change the area of a circle, the radius also changes. The formula (1.1) may then be rewritten to look like this:

$$r = \sqrt{\frac{A}{\pi}} \,.$$

But choosing the radius as the independent variable seems the more natural choice, because when we draw circles we use the radius as reference—not the area.

[1] A quantity which does not vary but has a fixed value is called a *constant*.

[2] Here, it is important to note that in mathematics, we distinguish between upper and lower case letters—i.e. $x$ and $X$ er different variables.

[3] The number $\pi$, however, *does* have a fixed value, so it is not a variable, but a constant.

In many cases, the independent variable is chosen to be the one that makes the most sense. Other examples might be:

- When we calculate the air pressure $p$ at different altitudes $h$ on a mountain, $h$ is the independent variable. We can change the altitude by walking up or down the mountain, and the air pressure then changes according to our position. However, it would seem counter-intuitive to say that we change the air pressure, which then leads to changes in our altitude.
- The population $N$ in a city typically changes with the time $t$ (measured in years). Here $t$ is the independent variable. This is because the population changes when time passes—and not the changing population whichs affect the passing of time.

## 1.2 Representation

Relationships between variables may be described in a number of ways. We might represent the relationship by

- a table, where related values of two (or more) variables are shown side by side,
- a model, which is a formula or an equation describing the relationship, or
- a graph, which shows how the value of one variable depends on another.

Here, we will take a closer look at tables and models through a few examples. The graphical representation is described in a later section.

**Table 1.1:** The relationship between gas consumption and the price.

| Volume ($m^3$) | Price (DKK) |
|---|---|
| 10 | 1010.10 |
| 20 | 1082.70 |
| 30 | 1155.30 |
| 40 | 1227.90 |
| 50 | 1300.50 |
| 60 | 1373.10 |

**Example 1.1** A connection to a domestic gas supply costs DKK 937.50 annually, plus DKK 7.26 per $m^3$ of gas used.

We can describe this relationship between volume and price in a table showing the price for certain volumes of gas, see table 1.1.

But we can also express the relationship as a mathematical model:

$$P = 7.26 \cdot V + 937.50\ ,$$

where $P$ is the annual price, and $V$ is the consumed volume (in $m^3$) of gas.

In principle, a table only gives us information about a limited amount of data, whereas a model makes it possible to calculate any number of related values of the variables. If we look at the numbers in table 1.1, we can only know the price when the consumption has a certain value. The model enables us to calculate the price for any possible volume of gas.

Because a table only contains a limited amount of data, it is not always possible to translate a table directly into a model. If we have a theoretical description of the relationship (price per $m^3$ etc., as given in the example above), we may write down a model and compare it to the table.

If we do not know the theoretical relationship, we would have to analyse the data to find a model, which fits. Different methods exist for this purpose, one of them is described in chapter 3.

**Table 1.2:** The relationship between time $t$ and distance $s$ for a falling stone.

| $t$ (s) | $s$ (m) |
|---|---|
| 0 | 0 |
| 1 | 4.91 |
| 2 | 19.64 |
| 3 | 44.19 |
| 4 | 78.56 |

**Example 1.2** In an experiment, we let a stone drop from a great height and measure how far it has fallen after a given time. The measurements are listed in table 1.2.

A mathematical analysis of these data shows that the relationship may be described by the model

$$s = 4.91 \cdot t^2 \ ,$$

where $t$ is the time (in seconds) and $s$ is the distance fallen (in metres).[4]

[4]This relationship was determined experimentally by Galileo Galilei in the late 16th century.[3]

Notice that in the model in example 1.2, time is the independent variable while the distance is the dependent. This is because the stone travels as time passes—we cannot move the stone back and forth to change time.

## 1.3 Graphical Representation

If two variables are related, it is possible to draw a picture illustrating their relationship. Such a picture is called a *graph*, and it is drawn in a *coordinate system.*

### Coordinate Systems

A coordinate system is a sort of grid which we lay across the plane. It is used to describe the position of points.

We start by drawing two perpendicular axes. The first axis (often called the $x$-axis) from left to right and the second axis (often called the $y$-axis) pointing upwards. The two axes are actually number lines, which intersect at their respective 0's, see figure 1.3. The intersection between the two axes is called the *origin* of the coordinate system.

If we draw a line from a point in the plane perpendicular to the first axis, the line intersects this axis at a number. We call this number the first coordinate (or $x$-*coordinate*) of the point. Similarly, we define the second coordinate (or $y$-*coordinate*) to be the number we get when a line perpendicular to the second axis intersects this axis. Any point can be described by its coordinates, i.e. a point is a pair of coordinates $(x, y)$, which tell us, where the point is placed in the coordinate system (see figure 1.3). The origin has coordinates $(0, 0)$.

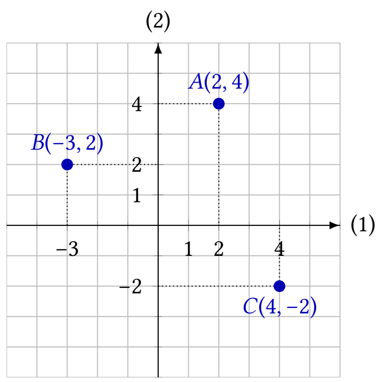
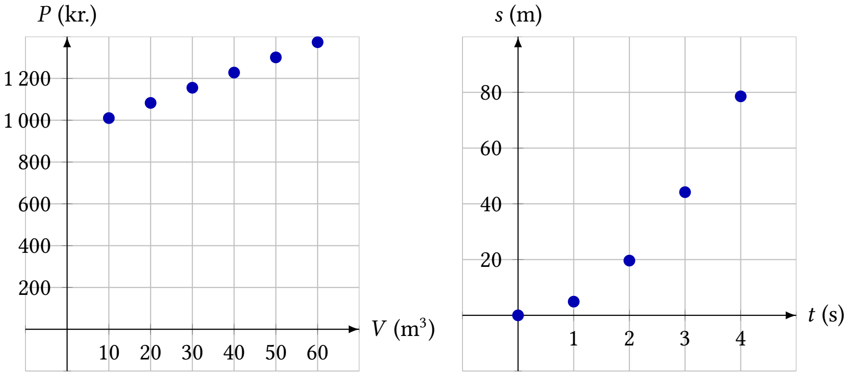



**Figure 1.3:** The three points $A(2, 4)$, $B(-3, 2)$ and $C(4, -2)$ drawn in a coordinate system.

### From Tables to Graphs

A table can be drawn in a coordinate system as a series of points by letting the first coordinate be the independent variable, and the second coordinate be the corresponding value of the dependent variable.

**Example 1.3** The tables from example 1.1 and 1.2 show the relationship between gas consumption and price, and between time and distance. The




(a) Consumption and price. (b) Time and distance.

**Figure 1.4:** Data from table 1.1 and 1.2 shown in separate coordinate systems. Both suggest a simple relationship between the variables—the points in the first coordinate system seem to fall on a straight line, whereas the points in the second seem to fall on a simple curve.

numbers in table 1.1 translate into a series of points:

$$(10, 1010.10)\ , \quad (20, 1082.70)\ , \quad (30, 1155.30)\ ,$$
$$(40, 1227.90)\ , \quad (50, 1300.50)\ , \quad (60, 1373.10)\ ,$$

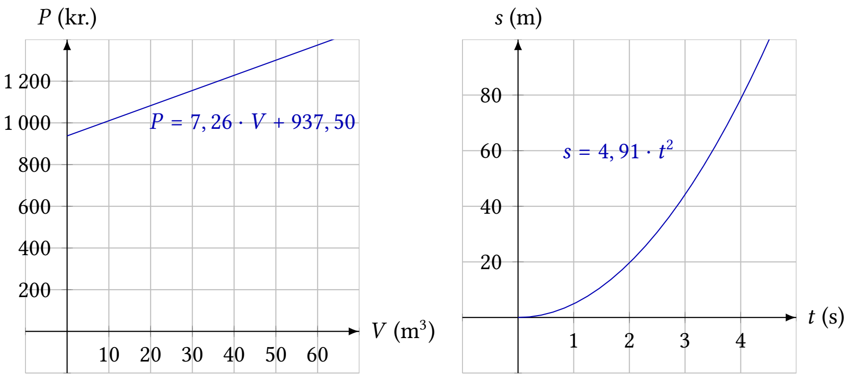
where the first coordinate is the consumption of gas, and the second coordinate is the price. We can then draw these points in a coordinate system, see figure 1.4(a).

In the same way, we can translate the data from table 1.2 to coordinates and draw the points in a coordinate system. This gives us figure 1.4(b).

In the first coordinate system (figure 1.4(a)), the points appear to form a straight line. This is not the case in the second coordinate system (figure 1.4(b)). However, the points appear to form a simple curve. In both cases, we might therefore assume that both cases may be described using a simple model.

The models behind the illustrations in figure 1.4 are shown in example 1.1 and 1.2. The two models may be described by their graphs, see figure 1.5. The models are the two equations

$$P = 7.26 \cdot V + 937.50 \quad \text{and} \quad s = 4.91 \cdot t^2\ .$$




(a) Consumption and price. (b) Time and distance.

**Figure 1.5:** Graphs of the two models from example 1.1 and 1.2.

The graphs of these two models consist of all of those points, whose first and second coordinates fit the corresponding equation.

So, a graph is a visual representation of an equation containing two variables. In principle, the graph and the equation contain the same information.

**Example 1.4** The equation $y = 5 - x^2$ describes a curve in a coordinate system, see figure 1.6. In the figure, we see that the point $(1, 4)$ lies on the graph. We can also see this from the equation, because if we insert $x = 1$, we get

$$y = 5 - 1^2 = 4 \ .$$

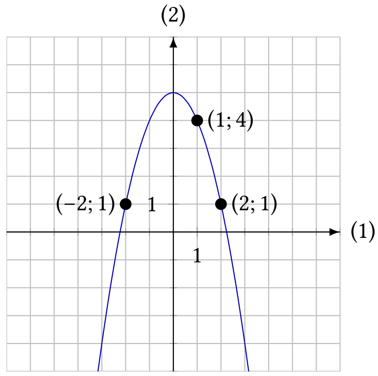


**Figure 1.6:** The graph of $y = 5 - x^2$.

We can also use the graph to solve the equation $5 - x^2 = 1$. This corresponds to letting $y = 1$ and finding the corresponding values of $x$. As the figure shows, there are two points on the graph where $y = 1$, i.e.

$$5 - x^2 = 1 \qquad \Leftrightarrow \qquad x = -2 \ \vee \ x = 2 \ .$$

We might also have found these two solutions by solving the equation algebraically.

## General Curves

In the previous sections, we have only looked at equations where the independent variable is isolated on the left hand side of the equation:

$$P = 7.26 \cdot V + 937.50 \ , \qquad s = 4.91 \cdot t^2 \quad \text{and} \quad y = 5 - x^2 \ .$$

But we might imagine equations, where it is not intuitive which variable is the independent and which is the dependent, e.g.

$$x \cdot y = 2 \quad \text{or} \quad y^2 - x = 3 \ .$$

These equations can also be translated into curves in a coordinate system. The curves will then consist of the points that fit these equations.

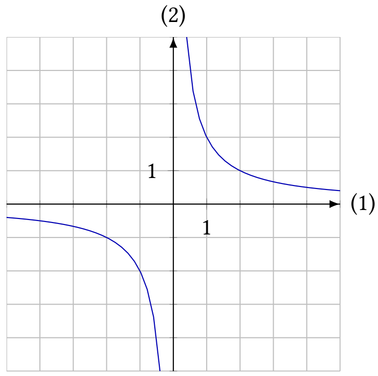


**Figure 1.7:** The curve with equation $x \cdot y = 2$.

**Example 1.5** The points on the curve $x \cdot y = 2$ are those points where the product of the first and second coordinates is 2. E.g.

$$(1, 2) \ , \quad (-2, -1) \quad \text{and} \quad (4, 0.5) \ .$$

If we draw this curve, we get the picture in figure 1.7.

There are no points on this curve with first coordinate 0. This is because letting $x = 0$ yields the equation

$$0 \cdot y = 2 \ ,$$

which has no solutions. For the same reason, there are no points on this curve with second coordinate 0.

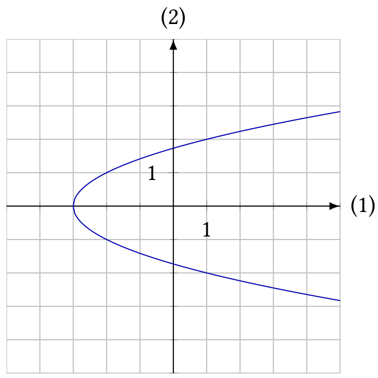


**Figure 1.8:** The curve with equation $y^2 - x = 3$.

**Example 1.6** The curve with equation $y^2 - x = 3$ is shown in figure 1.8.

This curve is not a graph. On this curve, we can find different points that have the same first coordinate, e.g. $(1, 2)$ and $(1, -2)$. On a graph, first coordinates must correspond to exactly *one* second coordinate.

## 1.4 Functions

A *function* in mathematics may be seen as a form of arithmerical operation. We may describe a function as a sort of machine which for any given input yields a certain output—so, a function is a relation between numbers.

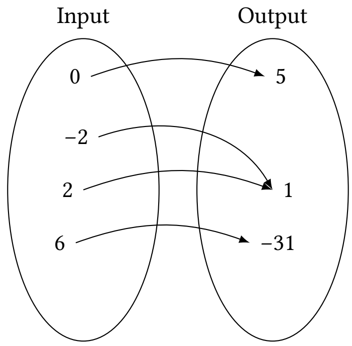



**Figure 1.9:** The function "square the number and subtract it from 5".

Figure 1.9 shows how the function "square the number and subtract it from 5" behaves. The output for 4 different numbers are shown. As we can see, different numbers may yield the same output—but it is not permitted to have different outputs from the same input.

Because it is impractical to describe functions in this way, a mathematical notation has been invented to make things easier. The function itself is denoted by a letter, e.g. $f$. Instead of "square the number, and subtract it from 5", we can now write $f$. The letter $f$ in itself contains no information on what the function $f$ does. If we want to show that, we need to write down a formula:

$$f(x) = 5 - x^2 \,.$$

This notation means that when we send a number ($x$) through the function $f$, what is done to the number is exactly what is described on the right hand side. $f(x)$ is read "$f$ of $x$", and the parenthesis shows that it is the number $x$, we send through the function. The number we get as output is called the *function value.*

**Example 1.7** Here, we look at the function $f(x) = x^2 - 3$.

The function values $f(-1)$ and $f(4)$ are calculated like this:

$$\begin{aligned} f(-1) &= (-1)^2 - 3 = 1 - 3 = -2 \\ f(4) &= 4^2 - 3 = 16 - 3 = 13 \,. \end{aligned}$$

It is important to note that the $x$ in the formula for $f(x)$ is a *place holder*, i.e it merely shows where we input numbers when we calculate the function value. Therefore, we may use other variables in place of $x$ instead of numbers—or we can replace $x$ by entire expressions.

**Example 1.8** Here, we look again at the function $f(x) = x^2 - 3$. But this time we calculate $f(t)$ and $f(x - 1)$, i.e. we insert $t$ and $x - 1$ in place of $x$:

$$\begin{aligned} f(t) &= t^2 - 3 \\ f(x-1) &= (x-1)^2 - 3 = x^2 + 1^2 - 2 \cdot x \cdot 1 - 3 = x^2 - 2x - 2 \,. \end{aligned}$$

If we input a mathematical expression, the output is another mathematical expression.

If we want to examine the behaviour of a function, it would be useful to draw its graph.

**Example 1.9** Here, we look at the function

$$f(x) = 3 \cdot \sqrt{x + 2} \,.$$

The graph of the function is the graph of the equation $y = f(x)$, i.e.

$$y = 3 \cdot \sqrt{x+2}\ ,$$

which is shown in figure 1.10.

We see in the figure that the graph passes through the point $(2, 6)$, i.e.

$$f(2) = 6\ .$$

We can also calculate this by inserting 2 in the formula for the function $f$:

$$f(2) = 3 \cdot \sqrt{2+2} = 3 \cdot \sqrt{4} = 3 \cdot 2 = 6\ .$$

But this just confirms what we see on the graph.

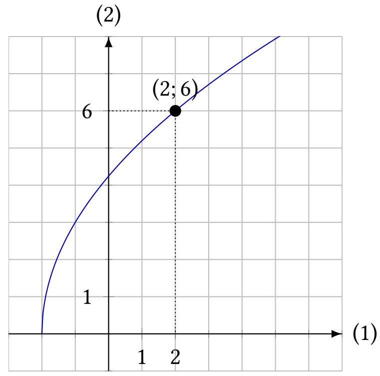



**Figure 1.10:** The graph of $f(x) = 3 \cdot \sqrt{x+2}$.

If, on the other hand, we know the function value, we can find the corresponding value of the independent variable. We can find it by using the graph, but we can also solve the problem by calculation, as in the following example:

**Example 1.10** Where does the function $g(x) = 2x + 1$ assume the value 17?

We find the answer to this question by solving the equation $g(x) = 17$. This can be done in the following way:

$$\begin{aligned} g(x) &= 17 && \Leftrightarrow \\ 2x + 1 &= 17 && \Leftrightarrow \\ 2x &= 16 && \Leftrightarrow \\ x &= 8\ . \end{aligned}$$

So the answer to the question is that $g(x) = 17$ when $x = 8$.

## Domain and Range

The function in example 1.9, $f(x) = 3 \cdot \sqrt{x+2}$, cannot use every possible value of $x$ as its input. E.g. if we try to calculate $f(-5)$, we get

$$f(-5) = 3 \cdot \sqrt{-5+2} = 3 \cdot \sqrt{-3}\ ,$$

but we cannot take the square root of negative numbers, so this calculation makes no sense. Therefore, this function value does not exist.

If we look at the graph of this function (figure 1.10), we see that the graph begins at $x = -2$. This is because the function value only makes sense for values of $x$ greather than or equal to $-2$. We say that the *domain* of $f$ consists of every number greater than or equal to $-2$. We can show this explicitly by writing this condition after the formula:

$$f(x) = 3 \cdot \sqrt{x+2}\ , \qquad x \geq -2\ .$$

Sometimes we state a domain consisting of fewer numbers than the ones that make sense mathematically. This might be because the function is a model which is restricted to certain numbers.

**Example 1.11** The population of a certain town during the years 2000–2017 is modelled by the function

$$b(t) = 24 \cdot t + 5309 \,, \quad 0 \le t \le 17 \,,$$

where $t$ is the number of years after 2000.

Here, $0 \le t \le 17$ shows that the domain is the numbers from 0 to 17. We restrict the domain to these numbers because the model only applies to the years 2000–2017. So, even if it is mathematically possible to calculate e.g. $b(-10)$ or $b(123)$, it is not allowed in this instance.

If we draw the graph, it therefore has to start at $t = 0$ and end at $t = 17$.

If we look again at the function

$$f(x) = 3 \cdot \sqrt{x+2} \,, \quad x \ge -2 \,,$$

we notice that the function values are never negative. The reason is that a square root is always positive. Therefore, the graph in figure 1.10 lies entirely above the first axis. The numbers which make up the possible function values are called the *range* of the function. In this case, the range of $f$ is the set of positive numbers.

When we look at a graph, the domain consists of all of the numbers on the first axis covered by the graph. The range consists of all the numbers on the second axis covered by the graph.

## 1.5 Proportionality

One of the important relationships variables might exhibit, is *proportionality*. There are two kinds of proportionality which are defined in the following way:

**Definition 1.12: Proportionality**

If $k$ is some constant ($k \neq 0$), we have:

1. $y$ is *directly proportional* to $x$ when $y = k \cdot x$.
2. $y$ is *inversely proportional* to $x$ when $y = \frac{k}{x}$.

The constant $k$ is called the *constant of proportionality.*

When we talk about direct proportionality, we often omit the word "directly". If we write "…is proportional to …", we mean "…is directly proportional to …".

If we rewrite the formulas in definition 1.12, the two types of proportinality may also be written as

1. $y$ is directly proportional to $x$ if $\frac{y}{x} = k$.
2. $y$ is inversely proportional to $x$ if $y \cdot x = k$.

Moreover, we see that when $y = k \cdot x$, $x = \frac{1}{k} \cdot y$. So, if $y$ is directly proportional to $x$ (with constant of proportionality $k$) then $y$ is directly proportional to $x$ (with constant of proportionality $\frac{1}{k}$).

When $y$ is inversely proportional to $x$, $y = \frac{k}{x}$, but then we also have $x = \frac{k}{y}$. Thus, if $y$ is inversely proportional to $x$, $x$ is also inversely proportional to $y$ (with the same constant of proportionality).

**Example 1.13** If we look at a mobile phone payment plan where texts are free, but calls cost DKK 0.70 per minute, the total cost will be directly proportional to the number of minutes used.

If we call the cost $C$ and the number of minutes $M$, we have:

$$C = 0.70 \cdot M \ .$$

Here, the constant of proportionality is 0.70.

**Example 1.14** If we drive from Odense to Copenhagen (a distance of approximately 160 km), the travel time will be inversely proportional to the speed with which we drive.

If the time $t$ is measured in hours and the speed $v$ is measured in kilometres per hour, the constant of proportionality will be 160, so

$$t = \frac{160}{v} \ .$$

From this relationship we gather (as we would expect) that if we drive at a speed of 80 km/h, the trip from Odense to Copenhagen will last 2 hours, whereas it will last only 1 hour if we drive at a speed of 160 km/h.

**Example 1.15** "$T$ is proportional to $p$ squared, and inversely proportional to $s$."

This relationship is expressed by the formula

$$T = k \cdot \frac{p^2}{s} \ ,$$

where $k$ is the constant of proportionality.

## 1.6 Intersections

If we analyse the graph of a function, several points on the graph may be of interest. For instance, we might be interested in those points where the graph intercepts the axes of the coordinate system.

Every point on the first axis has second coordinate 0. If the graph intercepts the first axis, the function value at the points of intercept must therefore be 0. We can then find the value of the first coordinate at these points by solving the equation $f(x) = 0$. On the second axis, every point has first coordinate 0. The second axis intercept can therefore be found by calculating the function value corresponding to $x = 0$, i.e. $f(0)$.

**Example 1.16** The function $f$ is given by

$$f(x) = -3x + 12\ .$$

The first axis intercept can then be found by solving

$$f(x) = 0 \quad \Longleftrightarrow \quad -3x + 12 = 0 \quad \Longleftrightarrow \quad x = 4\ .$$

So, the graph intercepts the first axis at $(4, 0)$.

The second axis intercept happens where

$$y = f(0) = -3 \cdot 0 + 12 = 12\ ,$$

so the graph intercepts the second axis at $(0, 12)$.

If we draw the graphs of two different functions, it is possible for these two graphs to intersect. We can then find their intersection by equating the formulas for the two functions and solving the equation.

This must be true, because at the intersection, both functions must share the same values of the independent variable and of the function value.

**Example 1.17** The two functions

$$f(x) = x - 5 \qquad \text{and} \qquad g(x) = -2x + 1$$

have intersecting graphs (see figure 1.11).

We find the coordinates of the intersection point by solving the equation $f(x) = g(x)$. We get

$$\begin{aligned} x - 5 &= -2x + 1 && \Longleftrightarrow \\ 3x &= 6 && \Longleftrightarrow \\ x &= 2\ . \end{aligned}$$

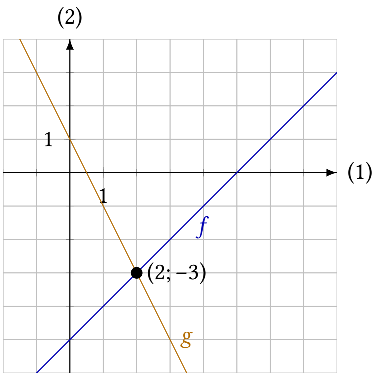


**Figure 1.11:** The intersection between the graphs of the two functions $f(x) = x - 5$ and $g(x) = -2x + 1$.

Now we know the first coordinate. To completely determine the point, we also need to know the second coordinate. We find this by inserting the calculated first coordinate into one of the functions:

$$y = f(5) = 2 - 5 = -3\ .$$

So, the two graphs intersect at $(2, -3)$. This is shown in figure 1.11.

## Solving Equations Graphically

We can find intersections between graphs by solving an equation. But then we can also solve equations by finding intersection points between graphs.

If we have a given equation, we can view the left and the right hand side as formulas of functions. Where the two formulas are equal (i.e. where the graphs intersect) we find the solution(s) to the equation.

**Example 1.18** The equation

$$x^2 - 3 = -2x$$

may be solved by drawing the graphs of

$$y = x^2 - 3 \qquad \text{and} \qquad y = -2x \; .$$

In figure 1.12, we see that the two graphs intersect at $(-3, 6)$ and $(1, -2)$. Therefore, the equation has two solutions:

$$x = -3 \qquad \vee \qquad x = 1 \; .$$

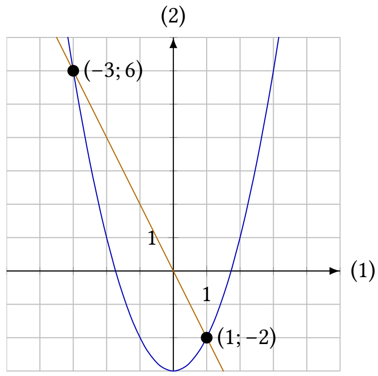



**Figure 1.12:** The solutions of $x^2 - 3 = -2x$ are the first coordinates of the intersection points.

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## 1.7 Exercises

**Exercise 1.1**
Here are three important terms when dealing with mathematical relationships. Describe the meaning of each of the terms.

a) Independent variable
b) Dependent variable
c) Constant

**Exercise 1.2**
The relationship between the area $A$ of a circle and its radius $r$ is given by the following formula

$$A = \pi \cdot r^2 \; .$$

a) Explain which quantities are *variables* and which are *constants.*
b) Which variable is the emphindependent variable in the formula?
c) Which variable is the emphdependent variable in the formula?

**Exercise 1.3**
The relationship between the two variables $x$ and $y$ is given by the formula

$$y = \frac{10}{1 + x} \; .$$

a) Write down a table with corresponding $x$- and $y$-values.
b) Use the table to sketch a graph.
c) Use a CAS to draw the graph, and compare this to your sketch.
d) Explain the graphical meaning of the constant 10.

**Exercise 1.4**
The relationship between the two variables $t$ and $P$ is given by the formula

$$P = \frac{15}{1 + t} + 20 \; .$$

a) Write down a table with corresponding $x$- and $y$-values.
b) Use the table to sketch a graph.
c) Use a CAS to draw the graph, and compare this to your sketch.
d) Explain the graphical meaning of the constants 15 and 20.

**Exercise 1.5**
A function has the formula $f(x) = 3x - 14$.

a) Determine the function values of $x = 2$ and $x = 10$.
b) Determine which value of $x$ has function value 78.

**Exercise 1.6**
A function is given by the formula

$$g(x) = \frac{x}{x - 2} \; .$$

a) Calculate $g(1)$
b) Calculate $g(-2)$.
c) Solve the equation $g(x) = 2$.

**Exercise 1.7**
For a function, any value of $x$ corresponds to exactly one value of $y$.

Which of these may be considered functions?

a) For any number $x$, we let $y$ be the numbers which divide $x$.

b) For any number $x$, we let $y$ be the square of $x$.

c) For any number $x$, we let $y$ be the numbers for which $y^2 = x$.

d) For any number $x$, we let $y$ be the numbers greater than $x$.

e) For any number $x$, we let $y$ be the numbers equal to $x$.

**Exercise 1.8**
"Translate" the following into formulas of functions.

a) Add four to the number.

b) Multiply the number by five, and add two.

c) Square the number, multiply by seven, and subtract one.

d) Add four to the number then take the square root.

e) Square the number and multiply by two. Then add four times the number and subtract five.

**Exercise 1.9**
Find the domains of each of the following functions:

a) $f(x) = 2x - 3$

b) $f(x) = \sqrt{x - 5}$

c) $f(x) = \dfrac{3}{x - 6}$

d) $f(x) = \dfrac{1}{x^2 + 1}$

e) $f(x) = \dfrac{2x}{\sqrt{x - 3}}$

f) $f(x) = \dfrac{1}{x^2 - 9}$

**Exercise 1.10**
Draw the graphs of the following functions and determine their ranges:

a) $f(x) = 2x - 1\,, \quad -4 \le x \le 7$

b) $g(x) = x^2 - 4x + 1\,, \quad 0 < x \le 12$

c) $h(x) = \sqrt{x^2 + 7}\,, \quad -3 < x < 3$

**Exercise 1.11**
Describe the relationship between $x$ and $y$ with a formula when

a) $y$ is directly proportional to $x$ with constant of proportionality 4.

b) $y$ is inversely proportional to $x$ with constant of proportionality 9.

**Exercise 1.12**
$T$ is directly proportional to $S$, and $S = 4$ when $T = 12$.

Write down a formula describing the relationship between $T$ and $S$.

**Exercise 1.13**
Fill out the missing parts of the table below when $y$ and $x$ are directly proportional.

| $x$: | 1 | 3 | $\square$ | 9 |
|---|---|---|---|---|
| $y$: | $\square$ | 18 | 42 | $\square$ |

**Exercise 1.14**
Fill out the missing parts of the table below when $y$ and $x$ are inversely proportional.

| $x$: | 1 | 4 | $\square$ | 16 |
|---|---|---|---|---|
| $y$: | $\square$ | $\square$ | 4 | 2 |

**Exercise 1.15**
The two functions $f$ and $g$ are given by

$$f(x) = 3x + 8$$
$$g(x) = 7x - 4$$

Determine the intersection point between the graphs of the two functions.

**Exercise 1.16**
Solve the following equations graphically:

a) $3x - 2 = 4x + 7$

b) $x^2 + 5 = x - 1$

c) $\dfrac{10}{x - 6} = x + 3$

# Linear Functions 2

Functions may be grouped by the form of their formulas.[1] E.g., the functions

$$f(x) = 3x + 2, \qquad g(x) = 7x - 5 \qquad \text{and} \qquad h(x) = -4x + 3$$

have formulas that follow the same pattern.

Functions which look like $f$, $g$ and $h$ are called *linear functions*.

[1]They can also be grouped by the form of their graphs, but this is essentially the same thing.

**Definition 2.1**

A *linear function* is a function of the form

$$f(x) = ax + b \ ,$$

where $a$ and $b$ are two numbers.

One of the reasons, the functions are called linear, is that their graphs are always straight lines (see figure 2.1).

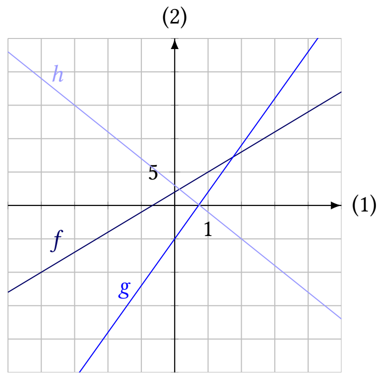


**Figure 2.1:** The graphs of the three linear functions $f$, $g$ and $h$.

## 2.1 Slope and Axis Intercepts

We can actually find the values of the numbers $a$ and $b$ in the formulas of the functions by looking at the graphs in figure 2.1. We have the following theorem:

**Theorem 2.2**

For a linear function $f(x) = ax + b$, the following holds:

1. If the independent variable $x$ increases by 1, the function value $f(x)$ increases by $a$.
2. The graph of the function intercepts the second axis at $b$.

**Proof**
When $x$ increases by 1, the function value increases from $f(x)$ to $f(x + 1)$. So, the function value increases by

$$\begin{aligned} f(x+1) - f(x) &= (a(x+1) + b) - (ax + b) \\ &= ax + a + b - ax - b \\ &= a \ . \end{aligned}$$

On the second axis, $x = 0$, i.e. the second axis intercept is

$$f(0) = a \cdot 0 + b = b\ . \qquad \blacksquare$$

For linear functions, the function value increases by a fixed number ($a$) whenever $x$ increases by 1. This is the reason why the graph is a straight line. The greater the number $a$, the faster $f(x)$ increases, and the line becomes steeper. The number $a$ is therefore called the *slope* of the line.

If $a$ is a negative number, $f(x)$ decreases when $x$ increases, and the line will slope downwards; then the function is decreasing.

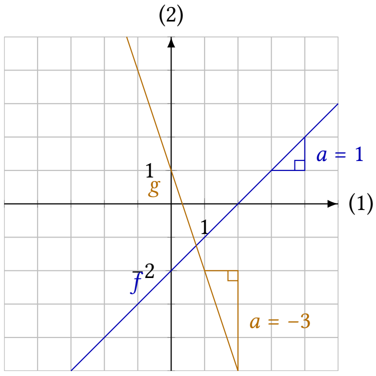



**Figure 2.2:** Determining the numbers $a$ and $b$.

**Theorem 2.3**

For a linear function $f(x) = ax + b$, the *slope* $a$ has the following properties:

1. If $a > 0$, the function is increasing.
2. If $a < 0$, the function is decreasing.

**Example 2.4** Figure 2.2 shows the graphs of two linear functions $f$ and $g$.

The graph of $f$ intercepts the second axis at $-2$, and if we move 1 to the right, we must move 1 up to stay on the graph, i.e. the slope is $a = 1$.

Therefore, $f$ has the formula $f(x) = 1 \cdot x + (-2)$ which simplifies to

$$f(x) = x - 2\ .$$

The graph of $g$ intercepts the second axis at 1, and when we move 1 along the first axis, the line moves 3 down the second axis, i.e. the slope is $-3$. The formula for $g$ is then

$$g(x) = -3x + 1\ .$$

In the special case where the slope of a linear function is 0, the function is constant; i.e. the graph is a straight line parallel to the first axis. Such a line has no intercepts with the first axis.[2]

[2] Unless the line lies entirely on the first axis, in which case they have an infinite amount of common points.

On the other hand, a linear function with a non-zero slope must have a graph which intercepts the first axis. We can calculate this intercept from the formula.

**Theorem 2.5**

The graph of the linear function $f(x) = ax + b$ intercepts the first axis at $\left(-\frac{b}{a}; 0\right)$.

**Proof**

On the first axis the second coordinate is 0.[3] The graph of $f$ therefore intercepts the first axis where

[3] Remember that every point on the first axis has the form $(x, 0)$, and every point on the second axis has the form $(0, y)$.

$$f(x) = 0\ ,$$

i.e.

$$ax + b = 0 \quad \Longleftrightarrow \quad ax = -b \quad \Longleftrightarrow \quad x = -\frac{b}{a} \,.$$

Thus, the graph intercepts the first axis at $\left(-\frac{b}{a}; 0\right)$. ∎

**Example 2.6** The linear function $f(x) = 4x - 12$ intercept the second axis at $b = -12$ and has slope $a = 4$. So, it intercepts the first axis at

$$x = -\frac{b}{a} = -\frac{-12}{4} = 3 \,.$$

Therefore, the graph intercepts the first axis at the point $(3, 0)$ and the second axis at the point $(0, -12)$.

## 2.2 Determining the Formula

If we know two points on the graph of a linear function, the function is clearly defined.[4] So, there must be a connection between the coordinates of the two points, and the two numbers $a$ and $b$.

[4]Exactly one straight line passes through any two given points.

This connection is given by a simple formula:

**Theorem 2.7**

If the graph of $f(x) = ax + b$ passes through the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then

$$a = \frac{y_2 - y_1}{x_2 - x_1} \,.$$

**Proof**

Figure 2.3 shows the graph of the function $f(x) = ax + b$, and the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

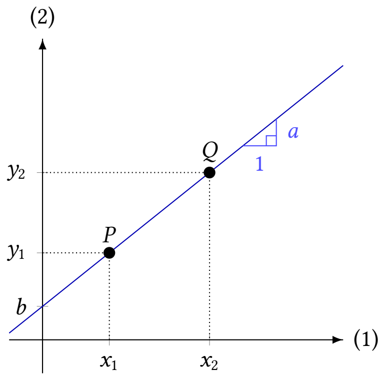



**Figure 2.3:** The two points $P$ and $Q$ on the graph of $f(x) = ax + b$.

The points $P$ and $Q$ are on the line, so $f(x_1) = y_1$ and $f(x_2) = y_2$. This gives us the equations:

$$\begin{aligned} y_2 &= ax_2 + b \,, \\ y_1 &= ax_1 + b \,. \end{aligned} \tag{2.1}$$

If we subtract the last equation from the first, we get

$$y_2 - y_1 = (ax_2 + b) - (ax_1 + b) \,,$$

which simplifies to

$$y_2 - y_1 = ax_2 - ax_1 \,.$$

Now, we factor out $a$ and get

$$y_2 - y_1 = a(x_2 - x_1) \quad \Longleftrightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = a \,,$$

which proves the formula. ∎

**Example 2.8** A linear function $f$ has a graph which passes through $P(3,5)$ and $Q(6,7)$. What is the formula for this function?

To answer this question, we look at the two points and see that

$$x_1 = 3\ , \quad y_1 = 5\ , \quad x_2 = 6 \quad \text{and} \quad y_2 = -7\ .$$

Now, we use the formula from theorem 2.7 and get

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-5}{6-3} = \frac{-12}{3} = -4\ .$$

So, the formula for the function is $f(x) = -4x + b$, and $b$ is yet unkown.

We then find the number $b$ by inserting the coordinates of one of the known points into the formula of the function. Since the graph of $f$ passes through $P(3,5)$, we know that $f(3) = 5$ and

$$\underbrace{-4\cdot 3 + b}_{f(3)} = 5 \qquad \Longleftrightarrow \qquad b = 5 + 4\cdot 3 \qquad \Longleftrightarrow \qquad b = 17\ .$$

Therefore, the formula of the function is $f(x) = -4x + 17$.

We can also deduce the following theorem, which we may use to determine a formula directly if the slope is known:

**Theorem 2.9**

If the linear function $f$ has slope $a$, and the graph of the function passes through the point $(x_1, y_1)$, then the formula of the function is given by

$$f(x) = a(x - x_1) + y_1\ .$$

**Proof**
A linear function has the general formula $f(x) = ax + b$. If the point $(x_1, y_1)$ is on the graph of $f$, then

$$y_1 = ax_1 + b \qquad \Longleftrightarrow \qquad b = y_1 - ax_1\ .$$

We insert this expression for $b$ into the formula of $f$:

$$f(x) = ax + (y_1 - ax_1)\ .$$

With a bit of calculation, we then get:

$$f(x) = ax + y_1 - ax_1 = ax - ax_1 + y_1 = a(x - x_1) + y_1\ ,$$

which proves the theorem. ■

**Example 2.10** If a linear function has a graph passing through the points $P(3,5)$ and $Q(6,-7)$, its slope is $a = -4$ (see example 2.8).

The formula of the function can then be found by inserting the coordinates of one of the two points in the formula from theorem 2.9. Here, we choose the point $Q(6,-7)$:

$$f(x) = a(x - x_1) + y_1 = -4(x-6) - 7 = -4x + 24 - 7 = -4x + 17\ .$$

So, in this way we find the same formula as we did in example 2.8.

## 2.3 Linear Growth

Linear functions grow in the following way:[5]

[5] We can also deduce this theorem from theorem 2.2.

**Theorem 2.11**

For a linear function $f(x) = ax + b$, whenever $x$ increases by $\Delta x$, the function value increases by $a \cdot \Delta x$.

**Proof**

If $x$ increases from $x_1$ to $x_2$, where $x_2 = x_1 + \Delta x$, the function value increases from

$$y_1 = f(x_1) = ax_1 + b$$

to

$$y_2 = f(x_2) = f(x_1 + \Delta x) = a(x_1 + \Delta x) + b = ax_1 + a \cdot \Delta x + b\ .$$

So, the function value increases by

$$y_2 - y_1 = (ax_1 + a \cdot \Delta x + b) - (ax_1 + b) = a \cdot \Delta x\ .$$

This proves the theorem. ∎

**Example 2.12** Table 2.4 shows an example of linear growth.

For the function $f(x) = 3x + 7$, every time $x$ increases by 2, the function value increases by $3 \cdot 2$.

**Table 2.4:** Growth of $f(x) = 3x + 7$.

| | $x$ | $y$ | |
|---|---|---|---|
| $+2$ | $-2$ | 1 | $+3 \cdot 2$ |
| $+2$ | 0 | 7 | $+3 \cdot 2$ |
| $+2$ | 2 | 13 | $+3 \cdot 2$ |
| | 4 | 19 | |

**Example 2.13** Here, we look at the function $f(x) = 3x - 4$ which has slope $a = 3$. If $x$ increases by $\Delta x = 5$, the function value increases by

$$a \cdot \Delta x = 3 \cdot 5 = 15\ .$$

Each time $x$ increases by 5, the function value increases by 15.

We could also ask how much $x$ must increase for the function value to increase by 60? In this case $a \cdot \Delta x = 60$, i.e.

$$3 \cdot \Delta x = 60 \qquad \Longleftrightarrow \qquad \Delta x = 20\ .$$

So, $x$ has to increase by 20 for the function value to increase by 60.

**Example 2.14** For the function $f(x) = -2x + 7$, whenever $x$ increases by $\Delta x = 3$, the function value increases by

$$a \cdot \Delta x = -2 \cdot 3 = -6\ .$$

When the function value increases by $-6$, it actually *decreases* by 6, each time $x$ increases by 3.[6]

[6] A negative increase corresponds to a decrease. In most mathematical growth models, it is useful to calculate using signs all the way, and then at the end use the sign of the result to determine whether we are looking at an increase or a decrease.

Next, we give a few examples of how a mathematical description of linear growth can answer different questions.

**Example 2.15** In a certain town, the population is given by

$$N(x) = 213x + 14\,752\ ,$$

where $N(x)$ is the population $x$ years after the year 2000.

The formula contains two constants,[7] 213 and 14 752. The function $N(x)$ is a linear function, so the number 213 is a slope: Each time $x$ increases by 1, the function value increases by 213. Since $x$ is measured in years and the function value is equal to the population, we see that the population increases by 213 inhabitants, whenever $x$ increases by 1 year. Therefore, the population increases by 213 inhabitants per year.

[7] Remember that a constant is a fixed number

14 752 is the second axis intercept. We find this number where $x = 0$. This happens in the year 2000,[8] and we therefore know that the population of the town was 14 752 in the year 2000.

[8] Since 2000 is 0 years after 2000.

**Example 2.16** Here we look at the same model as the one in example 2.15,

$$N(x) = 213x + 14.752\ .$$

How much does the population increase during a 10-year period?

The formula shows that the population increases by 213 inhabitants per year. During a 10-year period, we therefore have an increase of

$$10 \cdot 213 = 2130$$

inhabitants.

**Example 2.17** A company produces a number of items. The cost of production is a base cost of DKK 2000 and DKK 17 per item produced.

The total cost is therefore a function of the number of items produced. This function has the formula

$$o(x) = 17x + 2000\ ,$$

where $x$ is the number of items, and $o(x)$ is the total cost.

**Example 2.18** The mean temperature in West Greenland depends on the latitude[2] according to the model

$$T(x) = -0.732x + 46.1\ ,$$

where $T$ is the mean temperature (in $^\circ$C) and $x$ is the latitude.

So, the mean temperature in West Greenland decreases by 0.732$^\circ$C when the latitude increases by 1 degree.

If we try to interpret the number 46.1, we see that it should be the temperature at latitude 0 degrees, i.e. at the equator. But this interpreation is meaningless because the model only applies to West Greenland.

It is therefore impossible to give a physical interpretation of the number 46.1.

## 2.4 Piecewise Linear Functions

Functions whose graphs consist of pieces of straight lines are called *piecewise linear* functions. The graph of a piecewise linear function is shown in figure 2.5.

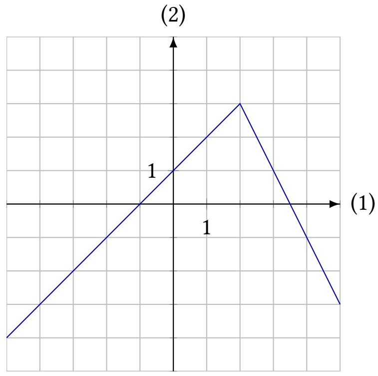



**Figure 2.5:** The graph of a piecewise linear function.

Here, we can see that when $x$ is less than 2, the graph corresponds to the equation

$$y = x + 1\,,$$

and when $x$ is greater than 2, the graph corresponds to the equation

$$y = -2x + 7\,,$$

If the function is denoted by $f$, we can write its formula like this:

$$f(x) = \begin{cases} x+1 & \text{for } x < 2 \\ -2x+7 & \text{for } x \geq 2 \end{cases}\,.$$

A function whose graph is an unbroken curve is called a *continuous* function. Piecewise linear functions are not necessarily continuous. An example of a *discontinuous* piecewise linear function might be

$$g(x) = \begin{cases} -x-4 & \text{for } x < -2 \\ 2x+2 & \text{for } -2 \leq x < 3 \\ \frac{1}{2}x+1 & \text{for } x \geq 3 \end{cases}\,.$$

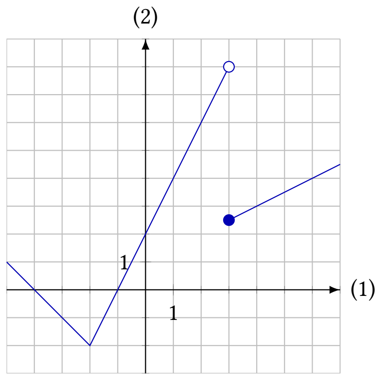



**Figure 2.6:** The graph of a discontinuous piecewise linear function.

The graph of $g$ is shown in figure 2.6.

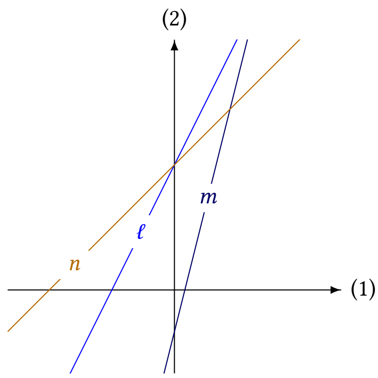
The filled circle on the graph marks a point which belongs to the graph, whereas the empty circle marks a point which does not belong to the graph. For the function $g$, the function value $g(3)$ must be calculated using the lower "branch" of the formula—therefore the point belongs to the rightmost line.

---

## 2.5 Exercises

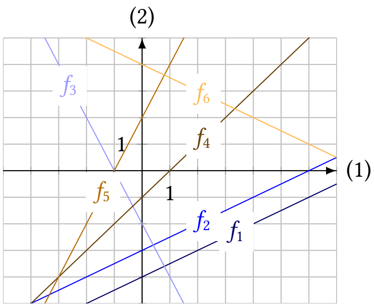
**Exercise 2.1**
The figure on the right shows the 3 straight lines $\ell$, $m$ and $n$ which are graphs of the functions

$$\begin{aligned} f(x) &= 4x - 1\,, \\ g(x) &= 2x + 3 \quad \text{and} \\ h(x) &= x + 3\,. \end{aligned}$$




Which line is the graph of which function?

**Exercise 2.2**
Use the graphs below to determine formulas for the functions $f_1, \ldots, f_6$.




**Exercise 2.3**
Determine a formula for the linear function whose graph passes through $P$ and has slope $a$:

a) $P(0, 4)$ og $a = 2$.

b) $P(2, 1)$ og $a = -\frac{1}{2}$.

**Exercise 2.4**
Determine a formula for the linear function whose graph passes through the points:

a) $A(2, 3)$ and $B(-1, 9)$

b) $C(-3, 2)$ and $D(-4, 1)$

c) $P(-5, 1)$ and $Q(7, 1)$

**Exercise 2.5**
The straight line $\ell$ passes through the points $A(4, -2)$ and $B(5, 5)$.

Calculate a formula for the linear function whose graph passes through $C(3, -2)$ and is parallel to $\ell$.

**Exercise 2.6**
A straight line $\ell$ passes through the points $P(3, -7)$ and $Q(8, 8)$.

a) Determine an equation for this line.

b) Determine where this line intercepts the first axis.

**Exercise 2.7**
Determine the intersection point between the graphs of the two linear functions

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = 3x + 7\ .$$

**Exercise 2.8**
For the linear function $h(t)$, $h(-1) = -13$ and $h(4) = 32$.

a) Determine a formula for this function.

b) Solve the equation $h(t) = 23$.

**Exercise 2.9**
A straight line with slope 3 passes through the point $(6, 14)$.

a) Determine an equation for this line.

b) What is the increase in $y$, when $x$ increases by 10?

**Exercise 2.10**
A straight line with slope 4 passes through the point $(2, 5)$.

a) Determine an equation for this line.

b) What is $\Delta y$ when $\Delta x = 12$?

**Exercise 2.11**
A straight line with slope $-2$ passes through the point $(3, 1)$.

a) Determine an equation for the line.

b) What is $\Delta x$ when $\Delta y = 38$?

**Exercise 2.12**
Determine by calculation whether the points $A$, $B$ and $C$ lie on a straight line when

a) $A(19, 12)$, $B(27, 25)$ and $C(40, 46)$

b) $A(-6, -3)$, $B(-1, 5)$ and $C(7, 18)$

**Exercise 2.13**
The price of a certain item grows linearly with time. The price was DKK 66 in 2010 and DKK 76 in 2015.

a) What will the price be in 2019?

b) When will the price be DKK 100?

**Exercise 2.14**
A linear function $f$ increases by 8 when $x$ increases by 4. It is also known that the graph of $f$ passes through the point $(3, 4)$.

Determine a formula for $f$.

**Exercise 2.15**
The function $f$ is given by

$$f(x) = \begin{cases} -x - 4 & \text{for } x \leq -1 \\ 2x - 1 & \text{for } x > -1 \end{cases}$$

a) Draw the graph of $f$.

b) Calculate $f(5)$ and $f(-2)$.

c) Solve the equation $f(x) = 6$.

d) Solve the equation $f(x) = -3$.

# 3 Linear Regression

When we measure a series of data, we might obtain a series of data points which only approximately form a straight line. An example is shown in figure 3.1.

Because the points do not lie exactly on a straight line, it would be wrong to use theorem 2.7 to calculate a formula. Depending on which two points we use, we would get different results for the values of $a$ and $b$.

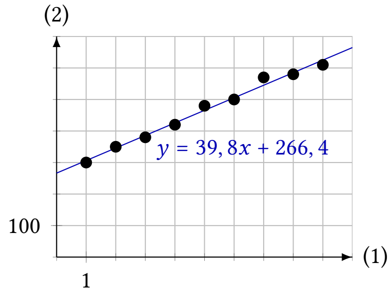



**Figure 3.1:** A series of data points and the straight line which is a best fit for these points.

Instead, we use a method called *linear regression* to determine which straight line is a best fit for all of the points. This method is built into most spreadsheets and mathematical computer programmes. Therefore, we usually only need to input the points in our programme which will then calculate the equation of the line. The equation in figure 3.1 is found in this way.

The idea is to find the line which minimises the distance from the line to all of the data points. This "collective distance" is defined to be the *sum of the squares* of the vertical distances from the line to the points. In figure 3.2, this sum is

$$D = d_1^2 + d_2^2 + d_3^2 + d_4^2 \ .$$

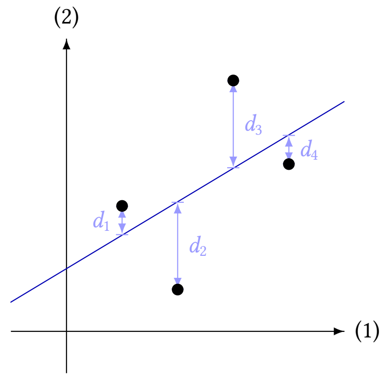



**Figure 3.2:** We minimise the sum of the squares $D = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

This sum contains as many terms as there are data points. The best fit is then the line which minimises $D$.

When we have $n$ data points with coordinates

$$(x_1; y_1), \dots, (x_n; y_n) \ ,$$

the slope of the line and its second axis intercept may be calculated from the $x$-values $x_1, \dots, x_n$ and the $y$-values $y_1, \dots, y_n$ using two formulas.

Quite advanced mathematics is required to deduce the two formulas for $a$ and $b$, so we are not going to prove them here; we merely state them in the following theorem:

**Theorem 3.1**

The best-fit straight line given the points $(x_1; y_1), \dots, (x_n; y_n)$ has equation $y = ax + b$ where

$$a = \frac{\overline{x \cdot y} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2}$$
$$b = \overline{y} - a \cdot \overline{x} \ .$$

Here, $\overline{x}$ is the mean of the $x$-values, $\overline{y}$ is the mean of the $y$-values, etc.

As previously stated, we are not going to prove these formulas. Instead, we provide an example which shows how to use them:

**Table 3.3:** Related values of $x$ and $y$.

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 6 |
| 6 | 8 |

**Table 3.4:** $x$, $y$, $x \cdot y$ and $x^2$. The mean values are listed in the last row.

| $x$ | $y$ | $x \cdot y$ | $x^2$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 2 | 3 | 6 | 4 |
| 4 | 6 | 24 | 16 |
| 6 | 8 | 48 | 36 |
| $\overline{x}$ | $\overline{y}$ | $\overline{x \cdot y}$ | $\overline{x^2}$ |
| 3 | 4.5 | 19.5 | 14 |

**Example 3.2** Table **??** shows the related values of the independent variable $x$ and the dependent variable $y$. The formulas for the best-fit straight line require a range of mean values. These are calculated in table 3.4.

We can now calculate

$$a = \frac{\overline{x \cdot y} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2} = \frac{19.5 - 3 \cdot 4.5}{14 - 3^2} = 1.2 \ .$$

and

$$b = \overline{y} - a \cdot \overline{x} = 4.5 - 1.2 \cdot 3 = 0.9 \ .$$

So, the best-fit straight line has the equation

$$y = 1,2x + 0,9 \ .$$

The points and the line are shown in figure 3.5.

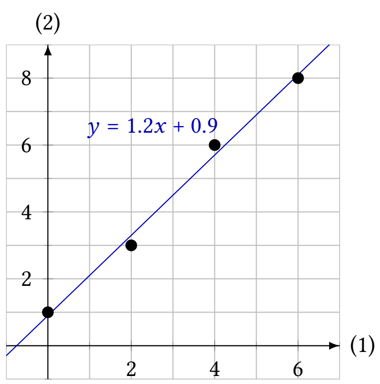



**Figure 3.5:** The 4 data points and the best-fit line.

The example demonstrates that using the formulas in theorem 3.1 to calculate an equation for the line can be quite cumbersome. So, in most cases we would simply use the tools built into a CAS to determine the best-fit line from the data points.

## 3.1 The Coefficient of Determination

When we use a CAS to find the best-fit line, we often also find a quantity known as the *coefficient of determination*, usually denoted $R^2$. It is a number between 0 and 1, which tells us how well the line fits the points. The closer this number is to 1, the better the line fits the points.

We can calculate the coefficient of determination by using the following formula:

**Theorem 3.3**

For the best-fit straight line of the points $(x_1, y_1), \dots, (x_n, y_n)$, the coefficient of determination is

$$R^2 = a^2 \cdot \frac{\overline{x^2} - \overline{x}^2}{\overline{y^2} - \overline{y}^2} .$$

The number satisfies $0 \leq R^2 \leq 1$, and the closer $R^2$ is to 1, the better the line fits the points.

**Example 3.4** If we look at the data set from example 3.2, we find

$$\overline{x} = 3 , \quad \overline{x^2} = 14 , \quad \overline{y} = 4.5 \quad \text{and} \quad \overline{y^2} = 27.5 .$$

In the example, we calculated $a = 1.2$, so the coefficient of determination is

$$R^2 = a^2 \cdot \frac{\overline{x^2} - \overline{x}^2}{\overline{y^2} - \overline{y}^2} = 1.2^2 \cdot \frac{14 - 3^2}{27.5 - 4.5^2} = 0.9931 .$$

This number is quite close to 1, therefore the straight line fits the data points quite well, which is also apparent from the graph in figure 3.5.

## 3.2 Why are Graphs Important?

When we find a straight line via linear regression, we do not actually need to draw the graph. We could just get a CAS to determine the equation of the line and the coefficient of determination $R^2$. We could then use the coefficient of determination to decide whether it is reasonable to use this straight line to model the data.

But it is always sensible to draw the graph; because it turns out that we can get the same straight line and coefficient of determination from widely different sets of data.

In 1973, the statistician Francis Anscombe described four different data sets which had the samme regression equation and coefficient of determination, but looked very different.[1] The four data sets are shown in table 3.6.

If we plot the data sets in four different coordinate systems, we get figure 3.7. Wee see clearly that the four data sets represent very different distributions of data. The first data set could quite possibly be modelled by a straight line—the points seem to be distributed randomly around the line. The next data set (top right) shows a definite relationship; but it is definitely not linear. The last to data sets both have a point which is positioned very differently from the rest of the data (such a point is called an *outlier*).

Despite these differences, all of the data sets have the same regression equation and determination coefficient, which are

$$y = 0.50 \cdot x + 3.00 , \quad R^2 = 0.67 .$$

We therefore conclude that drawing the graph is a good idea, because it allows us to determine how the points are distributed, before we decide whether or not to do a linear regression.

**Table 3.6:** Anscombe's four data sets. From [1].

| $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
|---|---|---|---|---|---|---|---|
| 4 | 4.26 | 4 | 3.1 | 4 | 5.39 | 8 | 6.58 |
| 5 | 5.68 | 5 | 4.74 | 5 | 5.73 | 8 | 5.76 |
| 6 | 7.24 | 6 | 6.13 | 6 | 6.08 | 8 | 7.71 |
| 7 | 4.82 | 7 | 7.26 | 7 | 6.42 | 8 | 8.84 |
| 8 | 6.95 | 8 | 8.14 | 8 | 6.77 | 8 | 8.47 |
| 9 | 8.81 | 9 | 8.77 | 9 | 7.11 | 8 | 7.04 |
| 10 | 8.04 | 10 | 9.14 | 10 | 7.46 | 8 | 5.25 |
| 11 | 8.33 | 11 | 9.26 | 11 | 7.81 | 8 | 5.56 |
| 12 | 10.84 | 12 | 9.13 | 12 | 8.15 | 8 | 7.91 |
| 13 | 7.58 | 13 | 8.74 | 13 | 12.74 | 8 | 6.89 |
| 14 | 9.96 | 14 | 8.1 | 14 | 8.84 | 19 | 12.5 |

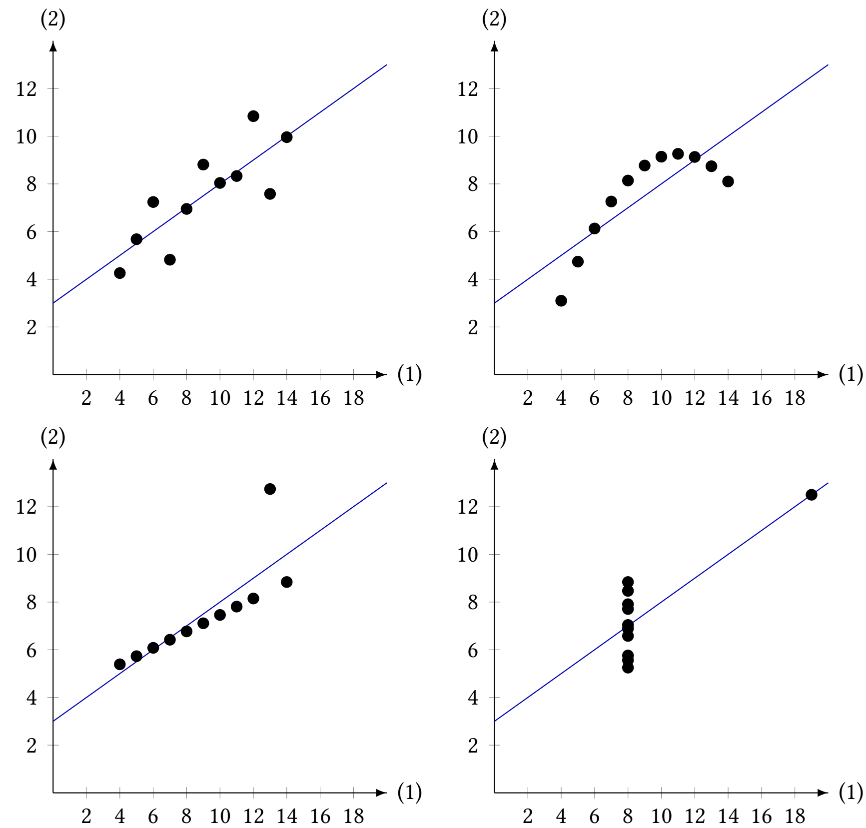



**Figure 3.7:** Anscombes four data sets plotted in four coordinate systems.
We see clearly that the distributions are very different.

In cases where one point is very different from the others, we might choose to investigate this further. Might it, for instance, be the result of a measurement error?

## 3.3 Residual Plots

When we determine the best-fit straight line of a series of data points, the points are distributed around this line. Because the point $(x_i, y_i)$ lies close to but not on the the line, it does not satisfy the equation $y = ax + b$, but instead

$$y_i = ax_i + b + \varepsilon_i \ ,$$

where $\varepsilon_i$ is the vertical distance from the point to the line. In a way, $\varepsilon_i$ expresses the error we make when we use the equation of the line to model the relationship between the variables. We call this the *residual* of the point.

If we isolate $\varepsilon_i$ in the equation (3.3), we get

$$\varepsilon_i = y_i - ax_i - b \ .$$

It is relatively easy to show that the mean of the residuals is 0. This is left as an exercise for the reader.

If the straight line is a good model of the relationship in question, the residuals are small compared to the $y$-values, and they are distributed randomly. We can investigate whether this is the case by drawing a *residual plot* which is a plot of the residuals as a function of the $x$-values.

**Example 3.5** In example 3.2, we found the best-fit straight line of a series of related values of $x$ and $y$. The equation of this straight line was

$$y = 1.2x + 0.9 \ .$$

Using $a = 1.2$ og $b = 0.9$, we calculate the residuals. E.g. we have

$$\varepsilon_1 = y_1 - ax_1 - b = 1 - 1.2 \cdot 0 - 0.9 = 0.1 \ .$$

The points and the residuals are listed in table 3.8.

**Table 3.8:** Related values of $x$ and $y$, as well as the residuals $\varepsilon$.

| $x$ | $y$ | $\varepsilon$ |
|---|---|---|
| 0 | 1 | 0.1 |
| 2 | 3 | −0.3 |
| 4 | 6 | 0.3 |
| 6 | 8 | −0.1 |

The residual plot is shown in figure 3.9. The second axis of this plot shows that the residuals are small compared to the measured $y$-values. The plot also shows that the residuals are distributed randomly around the first axis. In this case, it therefore seems reasonable to use a linear model.

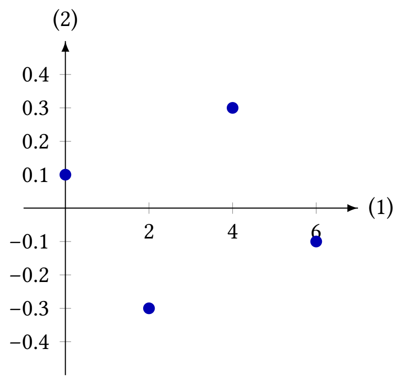



**Figure 3.9:** Residual plot of the values in table 3.8.

A series of data points might appear to lie on a straight line even though a different model describes the data much better. If this is the case, the residual plot typically displays some form of pattern.

**Example 3.6** Figure 3.10 shows a series of data points and a regression line. At a first glance, the line appears to be a good approximation to the points.

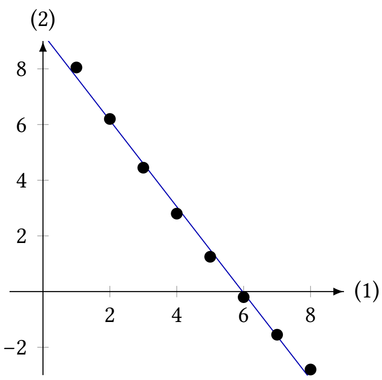



**Figure 3.10:** A series of data points and their regression line.

But when we look at the residual plot in figure 3.11, we see clearly that the straight line is not a good model for these points. In this case, the

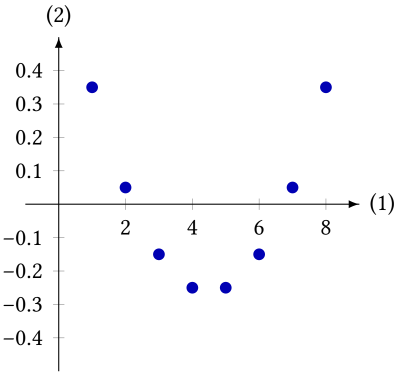



**Figure 3.11:** The residual plot of the data in figure 3.10.

residuals are not distributed randomly, but appear to form some sort of curve. Therefore, it would seem that the data should be described by some nonlinear model.

## 3.4 Exercises

**Exercise 3.1**
A series of related values of the variables $x$ and $y$ have been measured. The measurements are listed in the table below.

| $x$: | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$: | 5 | 7 | 8 | 11 |

a) Use the formulas in theorem 3.1 to calculate $a$ and $b$ for the best-fit straight line.

b) Calculate the residuals.

**Exercise 3.2**
The following 3 tables show 3 different relationships between the variables $x$ and $y$.

| $x$: | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$: | 3.2 | 4.9 | 7.3 | 8.8 | 10.9 | 13.4 |

| $x$: | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$: | 1.4 | 1.2 | 1.1 | 1.0 | 0.9 | 0.8 |

| $x$: | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$: | 0.1 | 1.4 | 2.9 | 4.6 | 6.5 | 8.6 |

a) Determine the best-fit line, and plot the residuals for each of the 3 data sets.

b) Use the residual plots to assess for which of the 3 data sets, the straight line is a good model.

**Exercise 3.3**
For a copper wire, the resistance measured in Ω depends linearly on the temperature measured in °C. A series of measurements have yielded the following data:

| Temperature (°C): | 0 | 15 | 30 | 45 | 60 |
|---|---|---|---|---|---|
| Resistance (Ω) : | 54.9 | 58.4 | 61.9 | 66.2 | 69.0 |

a) Determine the linear model which best fits the data.

b) At what temperature is the resistance 5 Ω?

**Exercise 3.4**
A series of measurements have yielded the following relationship between the wind speed and the noise produced by a windmill:

| Wind speed ($\frac{m}{s}$): | 6.3 | 7.2 | 8.5 | 9.4 |
|---|---|---|---|---|
| Noise level (dB) : | 51 | 56 | 65 | 71 |

The data set can be approximated by a linear model $y = ax + b$ where $x$ is the wind speed, and $y$ is the noise level.

a) Determine the numbers $a$ and $b$ in the equation.

b) What do the constant $a$ and $b$ represent in this model?

c) Determine the wind speed at which the noise level is 75 dB.

d) Determine how much the wind speed must decrease for the noise level to decrease by 10 dB.

# Bibliography


[1] F. J. Anscombe. “Graphs in Statistical Analysis”. In: *The American Statistician* 27.1 (Feb. 1973), pp. 17–21.

[2] Jesper Ruggaard Mehus and Svend Erik Nielsen. *Klimaforandringer i arktis.* Biofag nr. 6/2012. Særnummer. Forlaget Nucleus, 2012.

[3] Morton M. Sternheim and Joseph W. Kane. *General Physics.* 2nd edition. John Wiley & Sons, Inc., 1991.

## Fractions

| | | |
|---|---|---|
| Equivalent fractions | (1) | $\frac{a}{b} = \frac{a/k}{b/k}$ |
| | (2) | $\frac{a}{b} = \frac{k \cdot a}{k \cdot b}$ |
| Addition | (3) | $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d}$ |
| Subtraction | (4) | $\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d}$ |
| Multiplication | (5) | $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ |
| Division | (6) | $\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ |

## Powers and Roots

| | | |
|---|---|---|
| | (7) | $a^0 = 1$ |
| Negative exponent | (8) | $a^{-n} = \frac{1}{a^n}$ |
| Fractional exponent | (9) | $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ |
| Same base | (10) | $a^x \cdot a^y = a^{x+y}$ |
| | (11) | $\frac{a^x}{a^y} = a^{x-y}$ |
| | (12) | $(a^x)^y = a^{x \cdot y}$ |
| Same exponent | (13) | $a^x \cdot b^x = (a \cdot b)^x$ |
| | (14) | $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$ |
| Same root | (15) | $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{a \cdot b}$ |
| | (16) | $\frac{\sqrt[x]{a}}{\sqrt[x]{b}} = \sqrt[x]{\frac{a}{b}}$ |

## Algebra

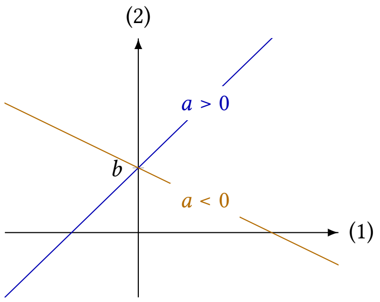
| | | |
|---|---|---|
| The associative law | (17) | $a + (b + c) = (a + b) + c = a + b + c$ |
| | (18) | $a(bc) = (ab)c = abc$ |

| | | |
|---|---|---|
| The distributive law | (19) | $a(b+c) = ab + ac$ |
| Square of a sum | (20) | $(a+b)^2 = a^2 + b^2 + 2ab$ |
| Square of a difference | (21) | $(a-b)^2 = a^2 + b^2 - 2ab$ |
| Difference of squares | (22) | $(a+b)(a-b) = a^2 - b^2$ |

## Functions

| | | |
|---|---|---|
| $y$ is directly proportional to $x$ | (23) | $y = k \cdot x$ |
| $y$ is inversely proportional to $x$ | (24) | $y = \frac{k}{x}$ |

## Linear functions




| | | |
|---|---|---|
| Linear function | (25) | $f(x) = ax + b$ |
| Slope from two points $(x_1; y_1)$ og $(x_2; y_2)$ on the graph | (26) | $a = \frac{y_2 - y_1}{x_2 - x_1}$ |
| $y$-axis intercept | (27) | $b = y_1 - ax_1$ |

## Linear regression

| | | |
|---|---|---|
| Best-fit straight line | (28) | $a = \frac{\overline{x \cdot y} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2}$ |
| | (29) | $b = \overline{y} - a \cdot \overline{x}$ |
| Coefficient of determination | (30) | $R^2 = a^2 \cdot \frac{\overline{x^2} - \overline{x}^2}{\overline{y^2} - \overline{y}^2}$ |
| Residual | (31) | $\varepsilon_i = y_i - ax_i - b$ |