Functions

Version 1.2
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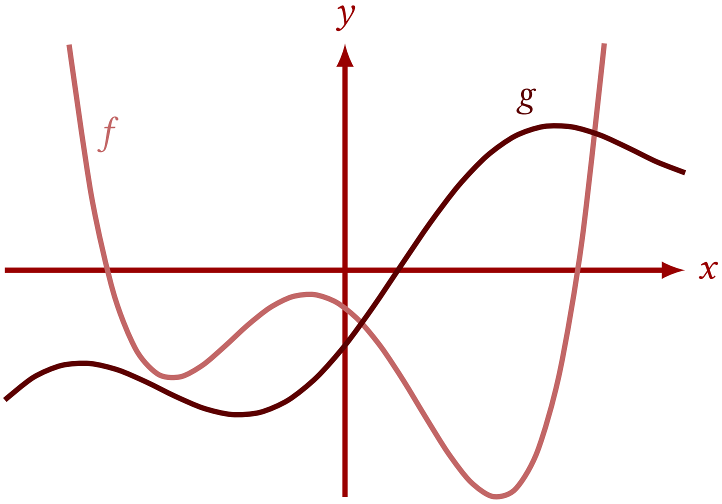

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Version 1.2, 2019

These notes are a translation of the Danish "Funktioner" written for the Danish stx.

The first chapter introduces the idea behind mathematical functions, and the following chapters give examples of certain types of functions. The appendix on set theory is included for reference when discussing domains and ranges of functions.

The document is written using the document preparation system LaTeX, see www.tug.org and www.miktex.org. Figures and diagrams are produced using *pgf/TikZ*, see www.ctan.org/pkg/pgf.

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What is a function? 1

In mathematics, a *function* is a formalisation of a relationship between variables. If the value of a variable y depends on the variable x, y is said to be *a function of* x. This means that to every possible value of x we assign exactly one value of y; the number we get when we send x through the function.

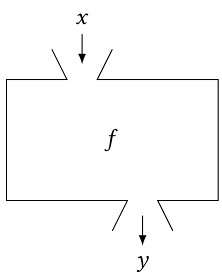


Figure 1.1: The function f interpreted as a kind of machine.

We may describe functions as a machine where x's are put in at one end, and the output at the other end of the machine are the y's (see figure 1.1).

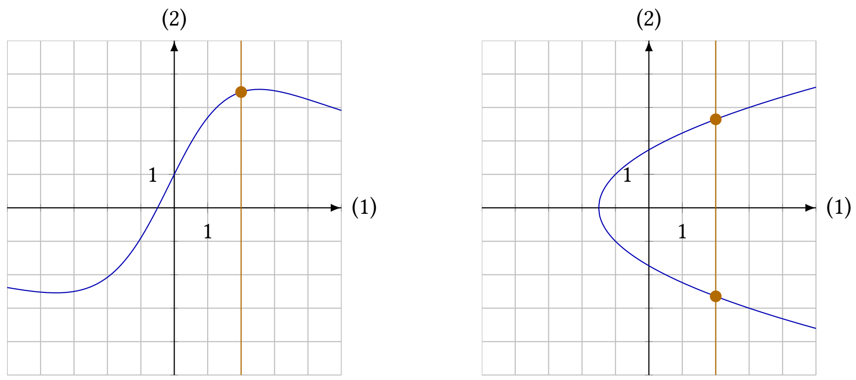
More formally, we say that a function is a relation between elements of one set X and the elements of another set Y, such that for each element $x \in X$ we have exactly one corresponding element $f(x) \in Y$. We say that $f(x)$ is *the image* of the element x.[1]

[1] It is important to remember that the parenthesis in $f(x)$ does not denote multiplication, but instead denotes the element we get when we send x through the function f.

The easiest way to visualise a function is to draw its graph. The graph of a function is a curve containing all points (x, y) where $y = f(x)$, i.e. the set

$$G = \{(x, y) \mid x \in X \ \wedge \ y = f(x)\} \ .$$

We can describe a function this way, but we would often be more interested in drawing the graph in a coordinate system. We draw the graph by marking all the points in the set G. Since each value of x corresponds to one and only one value of y, we can look at a curve and immediately see if it is the graph of a function or not. If a given curve is the graph of a function, any vertical line in the coordinate system intersects the curve at most once (see figure 1.2).



(a) The graph of a function.

(b) Not the graph of a function.

Figure 1.2: A function assigns exactly one y-value to each x-value. Therefore, vertical lines intersect the graph at most once.

There is a unique relationship between a function and its graph. But it is seldom possible to visualise the entire graph of a function, since a coordinate system is infinite and a drawing is limited. Therefore, if we want to know everything about a function, it would be useful to have a *formula* of the function, which shows us how to calculate the number $f(x)$ when we know the number x.

Example 1.1 We look at the function f, which has the formula

$$f(x) = \frac{x}{2} - \sqrt{x+1} .$$

Using the formula of the function, we can calculate every point on the graph. E.g., we have

$$f(8) = \frac{8}{2} - \sqrt{8+1} = 4 + 3 = 7 ,$$

and

$$f(24) = \frac{24}{2} - \sqrt{24+1} = 12 + 5 = 17 .$$

We now know that the points $(8, 7)$ and $(24, 17)$ are on the graph of this function.

1.1 Domain and range

The sets X and Y mentioned above describe the possible x-values and the possible y-values. The set X containing the possible x-values of the function f is also known as the function's *domain.*

Definition 1.2

Given a function f, the set of numbers x for which $f(x)$ exists, is called the *domain* of f.

We denote the domain of f by $\mathrm{Dom}(f)$.

A lot of the functions we are going to describe below have the entire set of real numbers as their domain; but there are cases where some numbers cannot be used as x-values.

Example 1.3 The function f has the formula

$$f(x) = \sqrt{x+3} .$$

Since we cannot take the square root of a negative number, $x + 3$ must always be positive, i.e.

$$x + 3 \geq 0 \quad \Longleftrightarrow \quad x \geq -3 .$$

Therefore, the domain of f contains every number greater than or equal to -3:

$$\mathrm{Dom}(f) = [-3; \infty[\ .$$

Example 1.4 The function g is defined by

$$g(x) = \frac{3}{4-x} \ .$$

We cannot divide by 0, wherefore the denominator of this fraction cannot be 0, i.e.

$$4 - x \neq 0 \quad \Longleftrightarrow \quad x \neq 4 \ .$$

So, the number 4 is not in the domain of g, and

$$\mathrm{Dom}(g) = \mathbb{R} \setminus \{4\} \ .$$

The set of all possible function values of a function is called the *range* of a function:

Definition 1.5

Given a function f, the set of possible function values of f is called the *range* of f. We denote this by $\mathrm{Ran}(f)$.

We have

$$\mathrm{Ran}(f) = \{f(x) \,|\, x \in \mathrm{Dom}(f)\} \ .$$

1.2 Combining functions

We can create new functions by using the ordinary mathematical operations. For example, we may multiply some function f by a number. If we multiply f by the constant c, we get the new function $c \cdot f$, which is defined in the following way:

$$(c \cdot f)(x) = c \cdot f(x) \ .$$

We may also add, subtract, multiply or divide two functions. We have the following definition:

Definition 1.6

Let two functions f and g, and a constant c be given. We then define the functions $c \cdot f, f + g, f - g, f \cdot g$ and $\frac{f}{g}$ to be the functions where

$$\begin{aligned} (c \cdot f)(x) &= c \cdot f(x) \\ (f + g)(x) &= f(x) + g(x) \\ (f - g)(x) &= f(x) - g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \end{aligned}$$

The definition states that we find the functions values of the new functions by performing the appropriate calculation on the functions values of the known functions. E.g., the function values of the function $c \cdot f$ are the function values of f multiplied by c. We must then have

$$\mathrm{Dom}(c \cdot f) = \mathrm{Dom}(f) \ ,$$

because $c \cdot f$ and f must be defined for the exact same values of x.

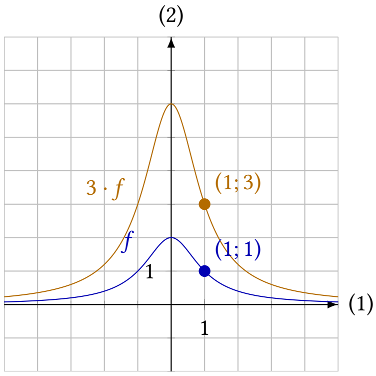


Figure 1.3: The graphs of f and $3 \cdot f$.

Example 1.7 Figure 1.3 shows the graphs of the functions f and $3 \cdot f$, where f is given by

$$f(x) = \frac{2}{x^2 + 1} .$$

The formula of $3 \cdot f$ is then

$$(3 \cdot f)(x) = 3 \cdot f(x) = \frac{6}{x^2 + 1} .$$

The figure shows that the graph of f passes through the point $(1, 1)$, while the graph of $3 \cdot f$ passes through the point $(1, 3)$. Thus, every function value is 3 times larger, and the graph of $3 \cdot f$ is the graph of f scaled by 3 in the vertical direction.

If we analyse the formula of f, we find

$$\mathrm{Ran}(f) = \ ]0; 2] \ .$$

Because every function value of $3 \cdot f$ is 3 times the corresponding function value of f, we have

$$\mathrm{Ran}(3 \cdot f) = \ ]0; 6] \ ,$$

which we can also see from the graph.

The example demonstrates that the graph of $c \cdot f$ is found by scaling the graph of f by a factor c. We can then find the range of $c \cdot f$ by multiplying every element of $\mathrm{Ran}(f)$ by c.

Example 1.8 For the two functions f and g given by

$$f(x) = x^2 - 5 \quad \text{and} \quad g(x) = 2^x ,$$

we have

$$f(3) = 3^2 - 5 = 4$$
$$g(3) = 2^3 = 8 ,$$

i.e.

$$(f \cdot g)(3) = f(3) \cdot g(3) = 4 \cdot 8 = 32 .$$

The functions described above are defined for all values of x where f and g are both defined. Therefore, we have

$$\mathrm{Dom}(f + g) = \mathrm{Dom}(f - g) = \mathrm{Dom}(f \cdot g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g) .$$

The exception is $\mathrm{Dom}\left(\frac{f}{g}\right)$, which cannot contain those elements where $g(x)$ is 0.[2] Here, we get

$$\mathrm{Dom}\left(\frac{f}{g}\right) = \mathrm{Dom}(f) \cap \{x \in \mathrm{Dom}(g) \,|\, g(x) \neq 0\} \ .$$

Determining the ranges of such functions require analysis of the specific situation.

[2] Since we are not allowed to divide by 0.

Example 1.9 Two functions f and g are given by

$$f(x) = \tfrac{1}{2}x \quad \text{and} \quad g(x) = \sqrt{x+4}\,.$$

The function $f+g$ has the formula

$$(f+g)(x) = \tfrac{1}{2}x + \sqrt{x+4}\,.$$

The graphs of all three functions are shown in figure 1.4.

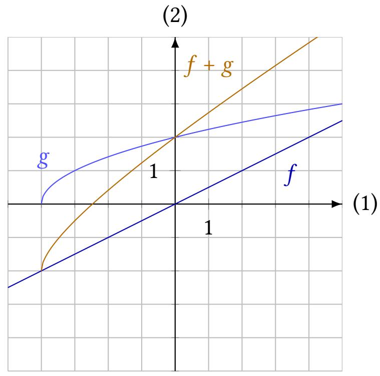


Figure 1.4: The graphs of f, g, and $f+g$.

The functions f and g have the domains

$$\mathrm{Dom}(f) = \mathbb{R} \quad \text{and} \quad \mathrm{Dom}(g) = [-4;\infty[\ .$$

The domain of $f+g$ is the intersection of these two sets:

$$\mathrm{Dom}(f+g) = [-4;\infty[\ .$$

We can use the graph to analyse the functions, and we find

$$\begin{aligned}\mathrm{Ran}(f) &= \mathbb{R}\\ \mathrm{Ran}(g) &= [0;\infty[\\ \mathrm{Ran}(f+g) &= [-2;\infty[\end{aligned}$$

Example 1.10 Figure 1.5 shows the graphs of the two functions

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2+1\,,$$

and the function $\frac{f}{g}$, which has the formula

$$\left(\frac{f}{g}\right)(x) = \frac{2x}{x^2+1}\ .$$

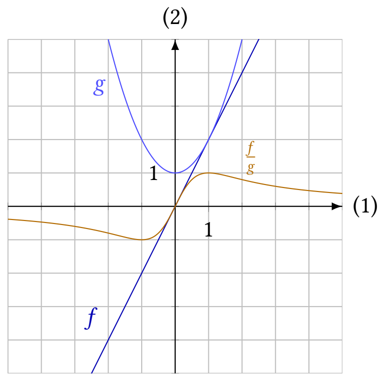


Figure 1.5: The graphs of f, g, and $\frac{f}{g}$.

The figure shows that

$$\mathrm{Ran}(f) = \mathbb{R} \quad \text{and} \quad \mathrm{Ran}(g) = [1;\infty[$$

while

$$\mathrm{Ran}\left(\frac{f}{g}\right) = [-1;1]\ .$$

Functions may also be combined by composition. We define the composite function $f\circ g$ in this way:

Definition 1.11: Composite function

Given two functions f and g, we define the composite function $f\circ g$ to be the function where

$$(f\circ g)(x) = f(g(x))\ .$$

So, we find the function values of $f\circ g$ by first calculating $g(x)$ and then using the function f on this number.

Example 1.12 The functions f and g are given by

$$f(x) = \sqrt{x+5} \quad \text{and} \quad g(x) = x^2 + 7 \ .$$

I.e.

$$g(2) = 2^2 + 7 = 11$$
$$f(11) = \sqrt{11+5} = 4 \ ,$$

so,

$$(f \circ g)(2) = f(g(2)) = f(11) = 4 \ .$$

We can find a formula for $f \circ g$ by replacing x in the formula for f by the formula for g. We then get

$$(f \circ g)(x) = \sqrt{g(x) + 5} = \sqrt{(x^2+7)+5} = \sqrt{x^2 + 12} \ .$$

The graphs of the functions f, g, and $f \circ g$ are shown in figure 1.6.

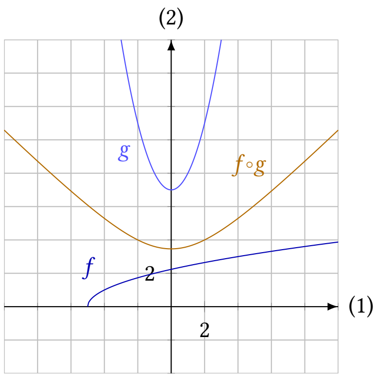


Figure 1.6: The graphs of f, g, and $f \circ g$.

1.3 Shifting graphs

If we shift the graph of a function, we get the graph of a new function. In this section, we describe a general method for finding the formula of the function we get when we shift a graph.

Figure 1.7 shows how a graph is shifted horizontally or vertically.

If the graph of the function $f(x)$ is shifted horizontally by x_0, we get the graph of a new function $g(x)$, where

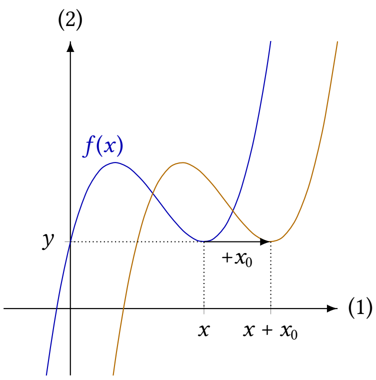
$$g(x + x_0) = f(x) \ .$$

We can rewrite this and get

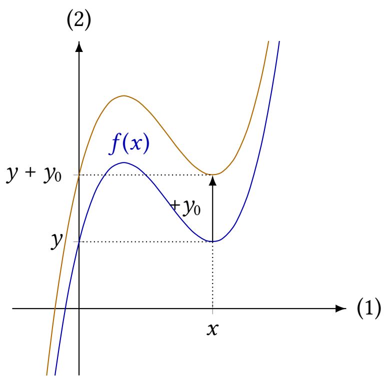
$$g(x) = f(x - x_0) \ ,$$

which we may use to find a formula for g if we know a formula for f.

Figure 1.7: We can shift a graph horizontally and vertically.



(a) Horizontal shift.



(b) Vertical shift.

If we shift the graph of f vertically by y_0, we get the graph of the new function g, which satisfies

$$g(x) = f(x) + y_0 \ .$$

When we shift a graph both horizontally and vertically, we say that we shift the graph by (x_0, y_0). Combining the results above, we get this theorem:

Theorem 1.13

If we shift the graph of the function $f(x)$ by (x_0, y_0), we get the graph of the function

$$g(x) = f(x - x_0) + y_0 \ .$$

Example 1.14 Figure 1.8 shows the graph of $f(x) = \sqrt{x}$ shifted by $(1, 3)$. According to theorem 1.13 this yields the graph of

$$g(x) = f(x - 1) + 3 = \sqrt{x - 1} + 3 \ .$$

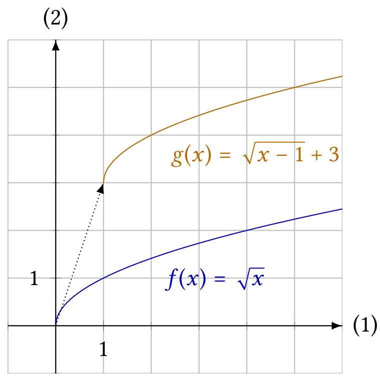


Figure 1.8: The graph of $f(x) = \sqrt{x}$ shifted by $(1, 3)$.

1.4 Inverse functions

A function is a unique mapping of one set of numbers (the domain) onto another set of numbers (the range). The inverse function is the function which swaps the direction of this mapping. If the function f maps x onto y, the inverse function f^{-1} maps y onto x:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Example 1.15 The function f is given by the formula

$$f(x) = 3x - 5 \ .$$

Some of the function values of f are

$$f(1) = 3 \cdot 1 - 5 = -2$$
$$f(6) = 3 \cdot 6 - 5 = 13$$

Since $f(1) = -2$ and $f(6) = 17$, we have

$$f^{-1}(-2) = 1 \quad \text{and} \quad f^{-1}(17) = 6 \ .$$

The function f maps 1 to -2, so f^{-1} maps -2 to 1, etc.

Not every function has an inverse function. The inverse function is—as the name implies—a function, i.e. $f^{-1}(x)$ has to be uniquely determined.

Example 1.16 The function f given by

$$f(x) = x^2$$

has no unique inverse function. This is because e.g.

$$f(-4) = (-4)^2 = 16$$
$$f(4) = 4^2 = 16$$

Because $f(-4)$ and $f(4)$ are both equal to 16, we cannot define uniquely $f^{-1}(16)$, which means an inverse function of f does not exist.

Therefore, an inverse function f^{-1} of f only exists when different x-values also have different y-values. Functions which satisfy this are called *injective*:

Definition 1.17

Let f be a function. If for every pair $x_1, x_2 \in \mathrm{Dom}(f)$, we have

$$x_1 \neq x_2 \quad \Longrightarrow \quad f(x_1) \neq f(x_2)$$

we say that f is an *injective function.*

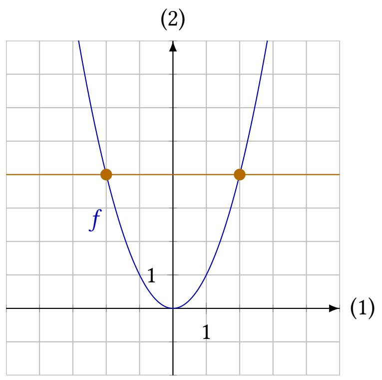


Figure 1.9: The function $f(x) = x^2$ is not injective.

We can investigate whether a certain function is injective by drawing its graph. If no two x-values have the same y-value, no horizontal line may intersect the graph more than once. As example 1.16 above shows, $f(x) = x^2$ is not injective (see figure 1.9).

So, injective functions have inverse functions. Formally, we may define them in this manner:

Definition 1.18

Let f be an injective function. The inverse function f^{-1} of f is the function satisfying

$$(f^{-1} \circ f)(x) = x \quad \text{and} \quad (f \circ f^{-1})(x) = x$$

for every $x \in \mathrm{Dom}(f)$.

The definition tells us that if we use the function f on x and then use f^{-1}, we get the element x back. I.e. f and f^{-1} are the exact opposites of each other. This means that if $y = f(x)$, then $x = f^{-1}(y)$.

Example 1.19 If the function f is given by the formula

$$f(x) = 2x - 6\ ,$$

The inverse function has the formula

$$f^{-1}(x) = \tfrac{1}{2}x + 3\ .$$

We can show that this is correct by determing formulas for $f^{-1} \circ f$ and $f \circ f^{-1}$. We get

$$(f^{-1} \circ f)(x) = \tfrac{1}{2}(2x - 6) + 3 = \tfrac{1}{2} \cdot 2x - \tfrac{1}{2} \cdot 6 + 3 = x - 3 + 3 = x$$

and

$$(f \circ f^{-1})(x) = 2 \cdot \left(\tfrac{1}{2}x + 3\right) - 6 = 2 \cdot \tfrac{1}{2}x + 2 \cdot 3 - 6 = x + 6 - 6 = x \,.$$

This shows that $f^{-1}(x) = \frac{1}{2}x + 3$ is the inverse function of f.

As we have previously demonstrated, $f(x) = x^2$ is not an injective function; but if we limit the domain to positive numbers, it is. In the next example, we demonstrate how to find the inverse function in this case.

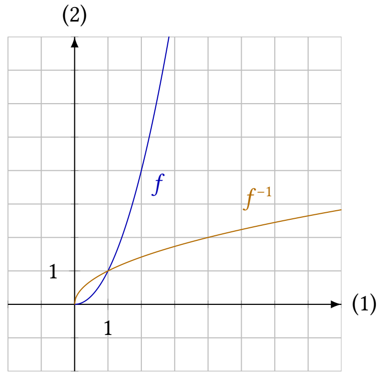


Figure 1.10: The function $f(x) = x^2$ $(x > 0)$ and its inverse $f^{-1}(x) = \sqrt{x}$.

Example 1.20 Let f be given by

$$f(x) = x^2 \,, \quad x \geq 0 \,.$$

The domain of f is the set $\mathrm{Dom}(f) = [0; \infty[$, and on this set, f is injective. The graph of $f(x)$ and its inverse function are shown in figure 1.10.

If $y = f(x)$, $x = f^{-1}(y)$. We can then find the inverse function of $f(x)$ by solving the equation $f(x) = y$ with respect to x:

$$f(x) = y \quad \Leftrightarrow \quad x^2 = y \quad \Leftrightarrow \quad x = \sqrt{y} \,.$$

I.e. we have $f^{-1}(y) = \sqrt{y}$, or

$$f^{-1}(x) = \sqrt{x} \,.$$

As the example above shows, we can find the inverse function $f^{-1}(y)$ by solving the equation $f(x) = y$. If we know $f^{-1}(y)$, we can then exchange variables if we would rather write the function as $f^{-1}(x)$.

Example 1.21 The function g is given by

$$g(x) = x^3 + 2 \,.$$

To determine the inverse function, we solve the equation

$$g(x) = y \quad \Leftrightarrow \quad x^3 + 2 = y \quad \Leftrightarrow \quad x^3 = y - 2 \quad \Leftrightarrow \quad x = \sqrt[3]{y - 2} \,.$$

I.e.

$$g^{-1}(x) = \sqrt[3]{x - 2} \,.$$

The graphs of the two functions g and g^{-1} are shown in figure 1.11

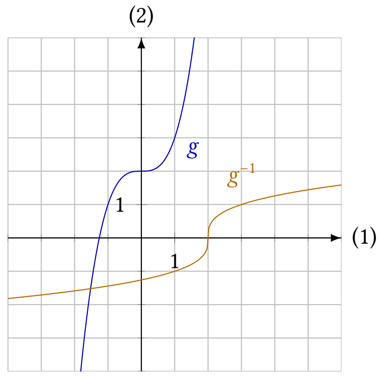


Figure 1.11: The function $g(x) = x^3 + 2$ and its inverse $g^{-1}(x) = \sqrt[3]{x - 2}$.

As figures 1.10 and 1.11 show, we find the graph of an inverse function by reflecting the graph of the original function in the line $y = x$. The reason is that we find the graph of the inverse function by taking all of the points on the graph of the original function and exchanging the x- and y-coordinates—because f^{-1} is the exact opposite of f.

1.5 Growth

We are often interested in investigating the growth of a function, i.e. how the dependent variable changes when the independent variable increases or decreases. E.g. we may investigate what happens when the independent variable increases by 1, or when it is doubled.

Absolute and relative growth

When we talk about growth, there are two ways to describe this growth: As an absolute or as a relative growth. The absolute growth shows how much a given quantity has grown, i.e. how much larger it has become. The relative growth shows how much a quantity has grown *compared to its initial value.*

Definition 1.22

If a quantity x increases from x_1 to x_2, the *absolute growth* is

$$\Delta x = x_2 - x_1 \ ,$$

and the *relative growth* is

$$r_x = \frac{\Delta x}{x_1} \ .$$

The absolute growth is the difference between the final value and the initial value, while the relative growth is the ratio of this difference and the initial value. We may calculate the relative growth r_x in different ways, since

$$\frac{\Delta x}{x_1} = \frac{x_2 - x_1}{x_1} = \frac{x_2}{x_1} - 1 \ .$$

Example 1.23 A quantity x increases from $x_1 = 5$ to $x_2 = 8$. The absolute growth is then

$$\Delta x = x_2 - x_1 = 8 - 5 = 3 \ ,$$

and the relative growth is

$$r_x = \frac{\Delta x}{x_1} = \frac{3}{5} = 0.6 \ .$$

The absolute growth shows that the quantity has increased by 3, while the relative growth shows that the quantity has increased by 0.6 times the initial value.

Note also that when we calculate the absolute growth, we always subtract the initial value from the final value, i.e. the difference is calculated *with sign.*

Example 1.24 A quantity t changes from $t_1 = 20$ til $t_2 = 14$. The absolute growth is then

$$\Delta t = 14 - 20 = -6 \ ,$$

and the relative growth is

$$r_t = \frac{\Delta t}{t_1} = \frac{-6}{20} = -0.3 \ .$$

Here, the absolute and the relative growth are both negative. The reason for this is that the quantity t has decreased. I.e. a growth of -6 shows us that the quantity has decreased by 6; and a relative growth of -0.3 shows us that the quantity has decreased by 0.3 times the initial value.

Relative growths are often written as a percentage. We find the percentage by multiplying the relative growth by 100 and writing the symbol % to indicate that we have done so. It may be easier to compare percentages, because the numbers are not as small as when we write the relative growths directly.

Example 1.25 A quantity p increaes from $p_1 = 25$ to $p_2 = 32$. The relative growth is

$$r_p = \frac{32 - 25}{25} = 0.28 = 28\% \ .$$

We can then say either that p has a relative growth of 0.28, or that p has increased by 28%.

Rewriting the formulas in definition 1.22 yields the following theorem, which shows how to find the final value when we know the initial value and either the absolute or the relative growth:

Theorem 1.26

If the quantity x increases from x_1 to x_2, then

$$x_2 = x_1 + \Delta x$$

and

$$x_2 = (1 + r_x) \cdot x_1 \ .$$

If a quantity has a certain relative growth, the initial value is multiplied by $1 + r_x$. So, relative growth corresponds to multiplying by some number.

Example 1.27 The quantity x has an initial value of $x_1 = 80$ and increases by 17%. What is the final value x_2?

The relative growth is 17%, i.e. 0.17. We then have

$$x_2 = (1 + 0.17) \cdot 80 = 93.6 \ .$$

So, the final value is 93.6.

When we investigate functions, we often would like to know how the function value increases when x increases. We therefore define the *function growth* in this way:

Definition 1.28

Let a function f be given. If the indepent variable increases from x_1 to x_2, the *function growth* is

$$\Delta f = f(x_2) - f(x_1) \ .$$

Example 1.29 A function f is defined by

$$f(x) = 4x - 5 \ .$$

If x increases from $x_1 = 10$ to $x_2 = 13$, the absolute growth is

$$\Delta x = 13 - 10 = 3\ ,$$

and the corresponding function growth is

$$\Delta f = f(13) - f(10) = (4 \cdot 13 - 5) - (4 \cdot 10 - 5) = 12\ .$$

So, when x increases from 10 to 13, the function value increases by 12.

Increasing and decreasing functions

A function is called *increasing* when the function value always gets larger as the independent variable increases. It is called *decreasing* when the function value always gets smaller as the independent variable increases. We define increasing and decreasing function in this way:

Definition 1.30

Let a function f, and two numbers $x_1, x_2 \in \mathrm{Dom}(f)$ be given. If

$$x_2 \geq x_1 \quad \Longrightarrow \quad f(x_2) \geq f(x_1)\ ,$$

we say that the function is *increasing*, whereas we say that it is *decreasing* when

$$x_2 \geq x_1 \quad \Longrightarrow \quad f(x_2) \leq f(x_1)\ ,$$

Thus, the definition says that a function is increasing if we get a larger function value by choosing a larger x-value. Similarly, a function is decreasing when we get a smaller function value by choosing a larger x-value.

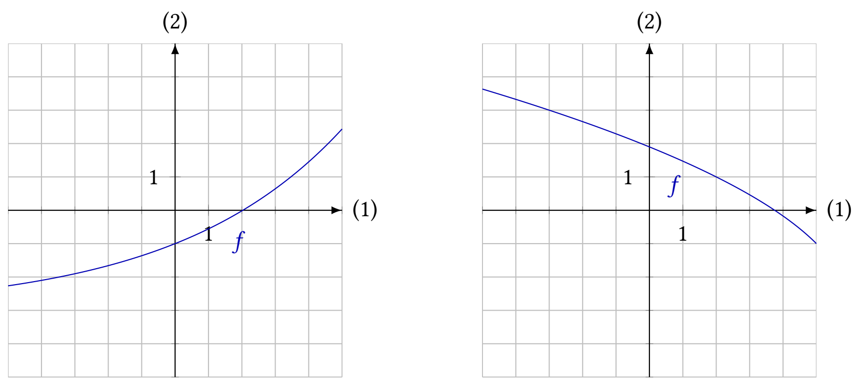
If $x_2 \geq x_1$, then $\Delta x \geq 0$. I.e. the conditions for a function to be increasing or decreasing may also be written as

$$\Delta x \geq 0 \quad \Longrightarrow \quad \Delta f \geq 0\ ,$$

and

$$\Delta x \geq 0 \quad \Longrightarrow \quad \Delta f \leq 0\ .$$

Figure 1.12: The graph of an increasing and a decreasing function. For increasing functions, the graph goes upwards as we move to the right, and for decreasing functions, the graph moves downwards as we move to the right.



(a) An increasing function. (b) A decreasing function.

Since an increasing function behaves in such a way that the function values become ever larger as x increases, the graph of an increasing function will move upwards to the right. The graph of a decreasing function will then move in the opposite direction, i.e. downwards to the right (see figure 1.12).

1.6 Exercises

Exercise 1.1
Determine the domains of the following functions:

a) $f_1(x) = \dfrac{x}{1-x}$ b) $f_2(x) = \sqrt{x+7}$

c) $f_3(x) = \dfrac{1}{x^2-9}$ d) $f_4(x) = 13$

e) $f_5(x) = \sqrt{4-x^2}$ f) $f_6(x) = \dfrac{7}{x^2+3}$.

Exercise 1.2
Draw the graphs of the following functions, and determine their ranges:

a) $f(x) = \sqrt{x}$

b) $g(x) = 3x - 1\,, \quad -4 \le x \le 7$

c) $h(x) = \dfrac{12}{x^2+3}$

d) $k(x) = 5 - \sqrt{x}$

e) $l(x) = \dfrac{1}{5 - \sqrt{x^2+9}}$

Exercise 1.3
The functions f and g are given by

$$f(x) = 3x - 6 \quad \text{and} \quad g(x) = x^2 - 9\,.$$

a) Calculate $(f+g)(2)$, $(f-g)(8)$, and $(f \cdot g)(0)$.

b) Determine formulas for $3 \cdot f$ and $\frac{f}{g}$.

c) Determine a formula for $3 \cdot f + \frac{g}{f}$.

d) Determine $\mathrm{Dom}\left(\frac{f}{g}\right)$ and $\mathrm{Dom}\left(\frac{g}{f}\right)$.

Exercise 1.4
Determine the ranges of $\frac{f}{g}$ and $\frac{g}{f}$ when

$$f(x) = 3x \quad \text{and} \quad g(x) = x^2 + 1\,.$$

Exercise 1.5
The two functions p and q are given by

$$p(x) = 4x - 2 \quad \text{and} \quad q(x) = -2x + 5\,.$$

Solve the equations

a) $(p+q)(x) = 5$ b) $(q-p)(x) = 13$

c) $(2p-q)(x) = 21$ d) $(p+3q)(x) = -7$

e) $\left(\frac{p}{q}\right)(x) = 6$ f) $\left(\frac{q}{p}\right)(x) = \dfrac{p(2)}{4}$

Exercise 1.6
Determine formulas for $f \circ g$ and $g \circ f$ when

a) $f(x) = x^2 - 3$ and $g(x) = 5x$

b) $f(x) = \dfrac{1}{x}$ and $g(x) = 1 + x$

c) $f(x) = \sqrt{9-x}$ and $g(x) = 2x - 3$

d) $f(x) = \sqrt{x^2+1}$ and $g(x) = \dfrac{x}{x-1}$

Exercise 1.7
The functions f and g are given by

$$f(x) = \sqrt{x+1} \quad \text{and} \quad g(x) = 3x^2 + 1\,.$$

a) Determine formulas for $f \circ g$ and $g \circ f$.

b) Determine the domains of $f \circ g$ and $g \circ f$.

c) Determine the ranges of $f \circ g$ and $g \circ f$.

Exercise 1.8
Determine a formula for the function whose graph is the graph of f shifted by $(4, 1)$ when

a) $f(x) = 5x - 3$ b) $f(x) = \sqrt{x+4}$

c) $f(x) = x^2$ d) $f(x) = \dfrac{x}{x+4}$

Exercise 1.9
Draw the graphs and use them to determine which of these functions are injective:

a) $f(x) = x^2 - 4$ b) $g(x) = \sqrt{2x + 10}$

c) $h(x) = \dfrac{1}{x + 3}$ d) $k(x) = \dfrac{x}{x^2 + 1}$

Exercise 1.10
Determine the inverse functions of

a) $f_1(x) = 2x + 6$ b) $f_2(x) = -x + 3$

c) $f_3(x) = x$ d) $f_4(x) = \frac{1}{2}x + 5$

e) $f_5(x) = 6x - 1$ f) $f_6(x) = -\frac{3}{4}x + \frac{1}{2}$

Exercise 1.11
The two functions f and g are given by

$$f(x) = \frac{x}{x - 2} \quad \text{and} \quad g(x) = \frac{2x}{x - 1} \ .$$

Are these functions each other's inverse functions?

Exercise 1.12
A quantity increases from $x_1 = 20$ to $x_2 = 37$.

Determine the absolute and the relative growth.

Exercise 1.13
A quantity t has an initial value of $t_1 = 45$. The quantity then increases with a relative growth of $r_t = 0.71$.

Determine the final value t_2.

Exercise 1.14
A quantity p increases from p_1 to p_2. The absolute growth is $\Delta p = 10$, and the relative growth is $r_p = 0.8$.

Determine p_1 and p_2.

Exercise 1.15
A function f is given by

$$f(x) = x^2 - x \ .$$

Determine the function growth when x increases from $x_1 = 5$ to $x_2 = 7$.

2 Linear functions

Linear functions are some of the simplest functions to work with. As you may recall, linear functions are defined in this way:

Definition 2.1

A *linear function* is a function of the form

$$f(x) = ax + b \ ,$$

where a and b are two numbers.

Here, we define linear functions based on their formulas. I.e. the definition does not contain any information on the properties of linear functions. Instead, these properties must be derived from the formula.

But it is possible to turn things around and base the definition of linear functions on the properties we want them to have. If we do so, we get this alternative definition of what a linear function is:

Definition 2.2: Alternative definition

A linear function f is a function where any fixed increase of the independent variable implies another fixed increase of the dependent variable.

If both definitions are possible, the definitions have to be equivalent. I.e. definition 2.1 must imply the property in definition 2.2, and the formula in definition 2.1 must be a consequence of the property in definition 2.2.

If we define linear functions according to definition 2.1, then the formula is given. We can then investigate the value of Δy for a fixed Δx. We get

$$\Delta y = f(x + \Delta x) - f(x) = (a(x + \Delta x) + b) - (ax + b) = a \cdot \Delta x \ .$$

I.e. given some Δx, $\Delta y = a \cdot \Delta x$. Therefore, a fixed absolute growth Δx of x leads to another (corresponding) fixed growth $\Delta y = a \cdot \Delta x$ of y. So, definition 2.1 does imply the property in definition 2.2.

If we, on the other hand, start with definition 2.2, we know that an absolute increase in x must lead to an absolute increase in y. The increase in y which correspnds to Δx, we call a. We then have

$$\Delta x = 1 \Rightarrow \Delta y = a \ .$$

Then we must also have

$$\Delta x = 2 \Longrightarrow \Delta y = 2a$$

and generally

$$()\Delta x = n \Longrightarrow \Delta y = n \cdot a) \quad \Longleftrightarrow \quad \frac{\Delta y}{\Delta x} = a\,. \tag{2.1}$$

If (x_0, y_0) is a known point on the graph of the function, and (x, y) is another arbitrary point, then $\Delta x = x - x_0$ and $\Delta y = y - y_0$, and we get

$$\frac{y - y_0}{x - x_0} = a \quad \Longleftrightarrow \quad y - y_0 = a \cdot (x - x_0)\,.$$

We can rewrite this equation and obtain

$$y = a \cdot (x - x_0) + y_0 = a \cdot x + y_0 - a \cdot x_0\,.$$

If we introduce the constant $b = y_0 - ax_0$, this is the formula from definition 2.1. So, definition 2.2 does lead to definition 2.1.

Therefore, both of these definition are equally valid, and we may define linear functions in different ways; depending on what we wish to emphasise.

Because $\Delta y = a \cdot \Delta x$, the number a describes how much Δy increases compared to Δx. If we look at the graph, a larger value of a will lead to a steeper graph.

We can even rewrite the equation 2.1 to get

$$a = \frac{\Delta y}{\Delta x} \quad \Longleftrightarrow \quad a = \frac{y_2 - y_1}{x_2 - x_1}\,,$$

which is the well-known formula for the slope of a linear function.

2.1 Exercises

Exercise 2.1
Determine, without drawing, a formula for the linear function, whose graph passes through the points:

a) $A(-1, 6)$ and $B(2, -3)$.
b) $P(2, 13)$ and $Q(-3, 3)$.
c) $R(6, -3)$ and $S(-1, 25)$.
d) $C(14, 4)$ and $D(6, 0)$.

Exercise 2.2
The linear functions f and g have parallel graphs. The graph of f passes through the point $A(3, 4)$ and intercepts the x-axis at $(-6, 0)$. The graph of g intercepts the y-axis at $(0, -3)$.

Determine formulas for each of the functions f and g.

3 Exponential functions

An *exponential function* is defined in the following way:[1]

> **Definition 3.1**
>
> An exponential function is a function of the form
>
> $$f(x) = b \cdot a^x \ ,$$
>
> where a and b are two positive numbers.

The number a in definition 3.1 is called the *multiplication factor* and b is called the *initial value*.

The reason b is called the initial value is that the graph of an exponential function intercepts the y-axis at the point $(0, b)$. This is because

$$f(0) = b \cdot a^0 = b \cdot 1 = b \ .$$

Examples of graphs of exponential functions are shown in figure 3.1. As the figure demonstrates, graphs of exponential functions do not intercept the x-axis. The reason is that the functions values cannot be negative; since a^x is always positive whenever a is a positive number—whatever the value of x may be.

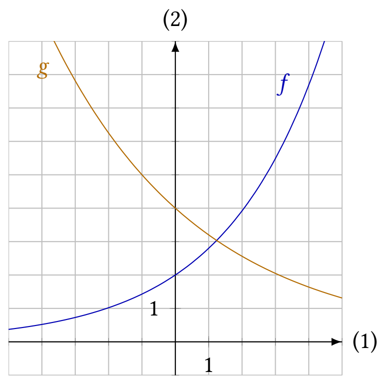


Figure 3.1: The graphs of the two exponential functions $f(x) = 2 \cdot 1.4^x$ and $g(x) = 4 \cdot 0.8^x$.

3.1 Exponential growth

Exponential functions increase in such a way, that every fixed absolute increase in the independent variable implies a fixed *relative* increase in the dependent variable. This is implied by the following theorem:

> **Theorem 3.2**
>
> Let $f(x) = b \cdot a^x$ be an exponential function. Whenever x increases by Δx, the function value is multiplied by $a^{\Delta x}$.

Proof

If x increases from x_1 to x_2, the function value increases from

$$y_1 = f(x_1) = b \cdot a^{x_1}$$

to

$$y_2 = f(x_2) = f(x_1 + \Delta x) = b \cdot a^{x_1+\Delta x} = b \cdot a^{x_1} \cdot a^{\Delta x} = y_1 \cdot a^{\Delta x} \ .$$

[1]In some contexts, only functions of the form $f(x) = a^x$ are called exponential functions, but here “exponential function” covers both cases.

Thus, the new function value y_2 is equal to $y_1 \cdot a^{\Delta x}$, and we have proved the theorem. ∎

Table 3.2: Growth of $f(x) = 4 \cdot 2^x$.

x	y
-3	0.5
0	4
3	32
6	256

Example 3.3 Table 3.2 shows the growth of an exponential function.

Here, we see that for the function $f(x) = 4 \cdot 2^x$, every time x increases by 3, the function value is multiplied by $2^3 = 8$.

When x increases by Δx, $f(x)$ is multiplied by $a^{\Delta x}$. I.e. the relative growth is

$$r_f = \frac{f(x_2) - f(x_1)}{f(x_1)} = \frac{a^{\Delta x} \cdot f(x_1) - f(x_1)}{x_1} = a^{\Delta x} - 1 \; .$$

So, the relative growth of the function equals $a^{\Delta x} - 1$.

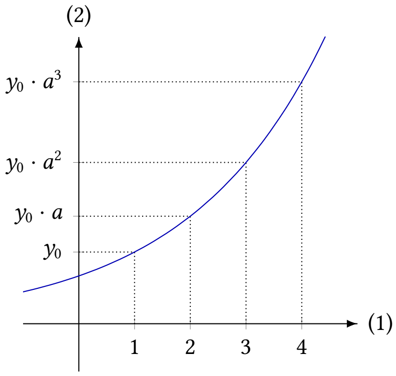


Figure 3.3: Growth of an exponential function.

Therefore, if $\Delta x = 1$, the function will have a relative growth of $a - 1$. If $a < 1$, this number is negative. This is an argument for the following theorem:

Theorem 3.4

For an exponential function $f(x) = b \cdot a^x$, we have:

1. If $a > 1$, the function is increasing.
2. If $0 < a < 1$, the function is decreasing.

As we have shown above, the relative growth $a - 1$ corresponds to an increase in x by 1, i.e. it makes sense to define the *growth rate* to be

$$r = a - 1 \; .$$

For the growth rate, we have the following theorem, which is a direct consequence of theorem 3.4.

Theorem 3.5

For an exponential function $f(x) = b \cdot a^x$, we define the *growth rate*

$$r = a - 1 \; .$$

We then have:

1. If $r > 0$, the function is increasing.
2. If $r < 0$, the function is decreasing.

When we know that exponential functions display relative growth, we can use them as mathematical models of cases which display this type of growth.

Example 3.6 In 2014, 8.6 million people lived in Honduras, and the population growth was 1.7% annually.[4] Therefore, we can describe the population of Honduras using an exponential function with an initial value of 8.6 and a growth rate of 1.7%.

The growth rate is 1.7%, wherefore the multiplication factor is

$$a = 1 + 1.7\% = 1 + 0.017 = 1.017 \ .$$

Therefore, the population is given by the function

$$f(x) = 8.6 \cdot 1.017^x \ ,$$

where x is the number of years after 2014, and $f(x)$ is the population in millions.

Example 3.7 A microbial culture grows exponentially, and the number of bacteria may be described by the function

$$B(t) = 364 \cdot 1.72^t \ ,$$

where t is the time in hours, and $B(t)$ is the number of bacteria.

From this formula, we derive the following: At the time $t = 0$, the culture contains 364 bacteria. The multiplication factor is 1.72, which means the growth reate is

$$r = 1.72 - 1 = 0.72 \ .$$

So, the number of bacteria have a relative growth of 0.72 per hour (or a growth of 72% per hour).

3.2 Determining a formula

If we know two points on the graph of an exponential function $f(x) = b \cdot a^x$, we can use the coordinates to determine the constants a and b (see figure 3.4).

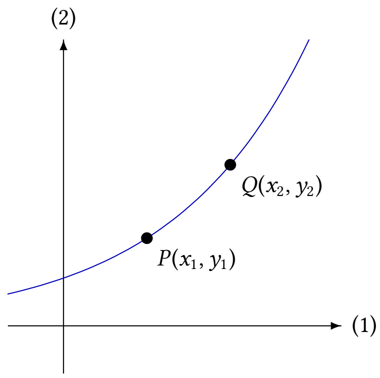


Figure 3.4: The graph of an exponential function passes through the points P and Q.

Theorem 3.8

If the graph of an exponential function $f(x) = b \cdot a^x$ passes through the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then

$$a = \sqrt[x_2 - x_1]{\frac{y_2}{y_1}} \qquad \text{and} \qquad b = \frac{y_1}{a^{x_1}} \ .$$

Proof
If $P(x_1, y_1)$ lies on the graph of $f(x) = b \cdot a^x$, we have

$$y_1 = b \cdot a^{x_1} \ . \tag{3.1}$$

The point $Q(x_2, y_2)$ also lies on the graph of f, so

$$y_2 = b \cdot a^{x_2}. \tag{3.2}$$

If we divide equation (3.2) by equation (3.1), we get

$$\begin{aligned} \frac{y_2}{y_1} &= \frac{b \cdot a^{x_2}}{b \cdot a^{x_1}} \qquad &\Leftrightarrow \\ \frac{y_2}{y_1} &= \frac{a^{x_2}}{a^{x_1}} \qquad &\Leftrightarrow \end{aligned}$$

$$\frac{y_2}{y_1} = a^{x_2 - x_1} \quad \Longleftrightarrow$$
$$\sqrt[x_2-x_1]{\frac{y_2}{y_1}} = a \ ,$$

and we have proven the formula for a.

To prove the formula for b, we isolate b in equation (3.1) and get

$$y_1 = b \cdot a^{x_1} \qquad \Longleftrightarrow \qquad \frac{y_1}{a^{x_1}} = b \ .$$

This proves the formula for b. ■

Example 3.9 If the graph of an exponential function $f(x) = b \cdot a^x$ passes through the two points $P(2, 12)$ and $Q(5, 96)$, then

$$x_1 = 2, \quad y_1 = 12, \quad x_2 = 5 \quad \text{and} \quad y_2 = 96 \ .$$

Using the formulas from theorem 3.8, we get

$$a = \sqrt[x_2-x_1]{\frac{y_2}{y_1}} = \sqrt[5-2]{\frac{96}{12}} = \sqrt[3]{8} = 2 \ ,$$
$$b = \frac{y_1}{a^{x_1}} = \frac{12}{2^2} = \frac{12}{4} = 3 \ .$$

Therefore, a formula for this function is $f(x) = 3 \cdot 2^x$.

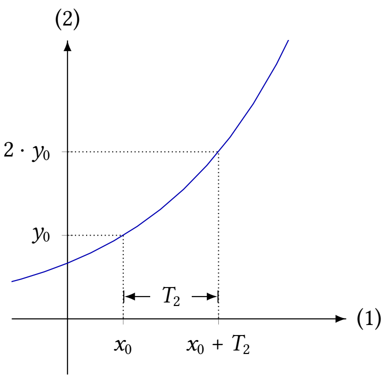


Figure 3.5: When we add T_2 to x_0, the function value doubles.

3.3 Doubling time and half-life

According to theorem 3.2, the function value of an exponential function has a fixed relative growth, when x has a fixed absolute growth. Therefore, if an exponential function is increasing, it makes sense to investigate how much x must increase for the function value to have a relative increase of 1—i.e. when it is doubled. We call this number the *doubling time* or the *doubling constant*, and denote it by T_2.

Theorem 3.2 states that the function value is multiplied by $a^{\Delta x}$ when x increases by Δx. Thus, to determine T_2 we need to find the value of Δx, so that $a^{\Delta x} = 2$.[2] So, T_2 is the solution to the equation

$$a^{T_2} = 2 \ .$$

The solution to this equation is[3]

$$T_2 = \frac{\log(2)}{\log(a)} \ .$$

Figure 3.5 illustrates the doubling time. An important point is that the function value doubles *every time* we add T_2 to x. Only exponential functions have this property.

When we look at decreasing exponential functions, it does not make sense to talk about doubling; instead, we determine the so-called *half-life*. The half-life $T_{\frac{1}{2}}$ is analogous to the doubling time, and we have the following theorem:

[2]Because doubling is the same as adding 100%.

[3]The log function is described in chapter 4.

Theorem 3.10

For an exponential function $f(x) = b \cdot a^x$, we have:

1. If f is increasing, the doubling time is $T_2 = \frac{\log(2)}{\log(a)}$.
2. If f is decreasing the half-life is $T_{\frac{1}{2}} = \frac{\log(\frac{1}{2})}{\log(a)}$.

Example 3.11 The exponential function $f(x) = 3 \cdot 1.7^x$ has a multiplication factor of $a = 1.7$. So, the doubling time is

$$T_2 = \frac{\log(2)}{\log(a)} = \frac{\log(2)}{\log(1.7)} = 1.31 \ .$$

I.e. each time x increases by 1.31, the function value doubles.

Therefore, an increase from $x = 5$ to $x = 6.31$ will double the function value, and so will an increase from $x = 100$ to $x = 101.31$.

3.4 Exercises

Exercise 3.1
An exponential function is given by $f(x) = 3.2 \cdot 1.7^x$.

a) By how much is the function value multiplied when the x-value increases by 1?
b) What is the relative growth of the y-value when the x-value increases by 1?
c) By how much is the function value multiplied if the x-value increases by 5?
d) What is the relative growth if the x-value increases by 5?
e) How much has the x-value increased if the function value has increased by 80%?

Exercise 3.2
An exponential function is given by $f(x) = 3.2 \cdot 0.63^x$.

a) By how much is the function value multiplied when the x-value increases by 1?
b) What is the relative growth of the y-value when the x-value increases by 1?
c) By how much is the function value multiplied if the x-value increases by 3?
d) What is the relative growth if the x-value increases by 3?
e) How much has the x-value increased if the function value has decreased by 50%?

Exercise 3.3
f is an exponential function, $f(x) = b \cdot a^x$. When $x = 2$ the function value is 10, and the function value has a relative growth of 0.25 when x has an absolute growth of 3.

Determine a formula for this function.

Exercise 3.4
The graph of an exponential function passes through the two points $(-2, 0.3)$ and $(5, 7.0)$

a) Determine a formula for this function.
b) Determine the growth rate.

Exercise 3.5
The graph of an exponential function $f(x) = b \cdot a^x$ passes through the points $(2, 6)$ and $(5, 45)$.

Determine a formula for this function.

Exercise 3.6
An exponential function $g(x)$ has a graph, which passes through the points $(1, 2)$ and $(3, 32)$.

a) Determine a formula for this function.
b) Determine $g(2)$.
c) Determine the relative function growth when x increases by 4.

Exercise 3.7
The graph of an exponentially increasing function passes through the points $(1, 6)$ and $(3, 54)$.

a) Determine a formula for this function.
b) Determine the doubling time.

Exercise 3.8
An increasing exponential function has a growth rate of 34.2%.

Determine the doubling time.

Exercise 3.9
A decreasing exponential function has a half-life of 8.9.

Determine the growth rate.

Exercise 3.10
A decreasing exponential function has a half-life of 6.5. The graph of this function passes through the point $(3.4, 20.9)$.

Determine a formula for this function.

Logarithms 4

An exponential equation is an equation of the form $a^x = k$, where a is the base and k is some number. We cannot solve these types of equations using the standard arithmetical operations. To solve these equations we need the so-called *logarithms*.

Definition 4.1

If a and k are two positive numbers, we define the number $\log_a(k)$ to be the number that solves the equation $a^x = k$.

We call the function $\log_a$ the *logarithm to base a*.

Example 4.2 From the definition of the logarithm to base a, we derive the following results:

$$\begin{aligned}
\log_2(8) &= 3 && \text{because} && 2^3 = 8\\
\log_7(49) &= 2 && \text{because} && 7^2 = 49\\
\log_{10}(10\,000) &= 4 && \text{because} && 10^4 = 10\,000\\
\log_9\left(\frac{1}{3}\right) &= -\frac{1}{2} && \text{because} && 9^{-\frac{1}{2}} = \frac{1}{3}\\
\log_{4.5}(2) &= 0.4608 && \text{because} && 4.5^{0.4608} = 2\,.
\end{aligned}$$

The definition states that if $x = \log_a(k)$, then x is the solution to the equation $a^x = k$. This implies that

$$a^{\log_a(k)} = k \quad \text{and} \quad x = \log_a(a^x)\,.$$

We may state this as a theorem:

Theorem 4.3

The logarithm to base a satisfies

$$a^{\log_a(x)} = x \quad \text{and} \quad \log_a(a^x) = x\,.$$

I.e. $\log_a(x)$ is actually the inverse function of a^x.

Rules for logarithms

Theorem 4.3 combined with exponentiation rules can be used to prove the following theorem:

Theorem 4.4

For the logarithm to base a we have:

1. $\log_a(r \cdot s) = \log_a(r) + \log_a(s)$.
2. $\log_a\left(\frac{r}{s}\right) = \log_a(r) - \log_a(s)$.
3. $\log_a(r^p) = p \cdot \log_a(r)$.

Proof
To prove the three rules, we use theorem 4.3, i.e. the fact that $r = a^{\log_a(r)}$ and $r = \log_a(a^r)$.[1]

1. For $\log_a(r \cdot s)$, we have

$$\log_a(r \cdot s) = \log_a\left(a^{\log_a(r)} \cdot a^{\log_a(s)}\right) = \log_a\left(a^{\log_a(r)+\log_a(s)}\right) = \log_a(r) + \log_a(s)\ .$$

2. For $\log_a\left(\frac{r}{s}\right)$, we have

$$\log_a\left(\frac{r}{s}\right) = \log_a\left(\frac{a^{\log_a(r)}}{a^{\log_a(s)}}\right) = \log_a\left(a^{\log_a(r)-\log_a(s)}\right) = \log_a(r) - \log_a(s)\ .$$

3. For $\log_a(r^q)$, we have

$$\log_a(r^p) = \log_a\left(\left(a^{\log_a(r)}\right)^p\right) = \log_a\left(a^{p \cdot \log_a(r)}\right) = p \cdot \log_a(r)\ .$$

This concludes the proof of the theorem. ∎

[1] In the proof we also use the identities

1. $a^n \cdot a^m = a^{m+n}$,
2. $\frac{a^n}{a^m} = a^{n-m}$ and
3. $(a^n)^m = a^{n \cdot m}$.

Example 4.5 We may derive from theorem 4.4 that

$$\log_a(ax) = \log_a(a) + \log_a(x) = 1 + \log_a(x)\ ,$$
$$\log_a\left(\frac{x}{a}\right) = \log_a(x) - \log_a(a) = \log_a(x) - 1\ ,$$
$$\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -1 \cdot \log_a(x) = -\log_a(x)\ .$$

It turns out that the connection between logarithms to different bases is quite simple:

Theorem 4.6

For the logarithm to base a and the logarithm to base b, we have

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}\ .$$

Proof
We may prove this theorem using the property in theorem 4.3 and rule 3 in theorem 4.4. We have

$$\log_a(x) = \log_a\left(b^{\log_b(x)}\right) = \log_a(b) \cdot \log_b(x)\ ,$$

i.e. $\log_a(x) = \log_a(b) \cdot \log_b(x)$, which we may rewrite as

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \ . \qquad \blacksquare$$

Theorem 4.6 shows that logarithms to different bases are proportional. It also shows that we actually only need one logarithmic function.

Example 4.7 If we want to calculate $\log_2(16)$, but only have a "$\log_{10}$-button" on our calculator, we may use theorem 4.6 to rewrite the calculation to a different base:

$$\log_2(16) = \frac{\log_{10}(16)}{\log_{10}(2)} \ .$$

We then calculate $\frac{\log_{10}(16)}{\log_{10}(2)}$ and get

$$\log_2(16) = 4 \ .$$

Because we only need one logarithmic function, we only need one logarithm button on a calculator. We then only need to agree upon which base, we would prefer to use. Usually, a calculator actually has two logarithms: one to base 10, and one to base e (Euler's number, which we describe in the next section).

The two logarithms $\log_{10}$ and $\log_{\mathrm{e}}$ are called the *common logarithm* and the *natural logarithm*, and are often denoted by log (the common logarithm) and ln (the natural logarithm), i.e.

- The *common logarithm* (log) is the logarithm to base 10; therefore $\log = \log_{10}$.
- The *natural logarithm* (ln) is the logarithm to base e; therefore $\ln = \log_{\mathrm{e}}$.

Therefore, we have

$$\begin{aligned} y = \log(x) \quad &\Longleftrightarrow \quad x = 10^y \quad \text{and} \\ y = \ln(x) \quad &\Longleftrightarrow \quad x = \mathrm{e}^y \ . \end{aligned}$$

Some CAS's use log to denote the natural logarithm instead of the common logarithm (this is also often the case in theoretical mathematics). Therefore, it makes sense to always use the natural logarithm (ln always denotes the natural logarithm); as theorem 4.6 demonstrates, we only need one logarithm.

4.1 The natural logarithm

Euler's number

The number e, sometimes called *Euler's number*, plays a major role in mathematics. Like π, it is an irrational number.[2] Thus e has an infinite amount of decimals. To a precision of 24 decimals,

$$\mathrm{e} = 2.718\,281\,828\,459\,045\,235\,360\,287\ldots \ .$$

[2] An irrational number is a number which cannot be written as a fraction. A characteristic of irrational numbers is that they have an infinite amount of decimals with no discernable pattern.

e turns up in a lot of different mathematical formulas, but here we will look only at the natural logarithm.

The natural logarithm

The natural logarithm is defined to be the logarithm to base e:

Definition 4.8

The *natural logarithm*, ln, is the logarithm to base e:

$$\ln(x) = \log_{\mathrm{e}}(x) .$$

Because logarithms are the inverses of exponential functions, we also have a "natural exponential function", which is the exponential function with base e. The natural exponential function is often denoted by exp, i.e.

$$\exp(x) = \mathrm{e}^x .$$

It turns out that it is possible to rewrite every exponential function so that they are based on this function. That is the topic of the next section.

4.2 Exponential functions

It is common to write an exponential function $f(x) = b \cdot a^x$ in another way. Because $a = \mathrm{e}^{\ln(a)}$, an exponential function may be written as

$$f(x) = b \cdot a^x = b \cdot \left(\mathrm{e}^{\ln(a)}\right)^x = b \cdot \mathrm{e}^{\ln(a) \cdot x} .$$

We may also write this as

$$f(x) = b \cdot \mathrm{e}^{kx} \quad (\text{where } k = \ln(a)) .$$

Example 4.9 The exponential function $f(x) = 4.6 \cdot 9.1^x$ may be written as

$$f(x) = 4.6 \cdot \mathrm{e}^{2.2x} ,$$

because $\ln(9.1) = 2.2$.

As we know, an exponential function $f(x) = b \cdot a^x$ is increasing when $a > 1$, and decreasing when $0 < a < 1$.

Since $k = \ln(a)$ this implies that exponential functions $f(x) = b \cdot \mathrm{e}^{kx}$ satisfy:

1. The function is increasing if $k > 0$.
2. The function is decreasing if $k < 0$.

Therefore, we sometimes distinguish between increasing and decreasing exponential functions, by separating them into two cases:

1. $f(x) = b \cdot \mathrm{e}^{kx}$ when the function is increasing.
2. $f(x) = b \cdot \mathrm{e}^{-kx}$ when the function is decreasing.

When we do this, k is always a positive number, and the sign is not viewed as a part of k.

There are several good reasons to write an exponential function in this way. One of them is that calculations involving units will make sense. Another good reason has to with a branch of mathematics known as calculus; however, the explanation will have to wait.

Doubling times and half-lives

For an increasing exponential function, we may calculate the doubling time T_2 using the formula

$$T_2 = \frac{\ln(2)}{\ln(a)} .$$

If the increasing exponential function has the form $f(x) = b \cdot e^{kx}$ this instead becomes[3]

$$T_2 = \frac{\ln(2)}{k} \qquad (\text{when } f(x) = b \cdot e^{kx}) .$$

[3] Here, we use that $k = \ln(a)$.

If a decreasing exponential function is written in the form $y = b \cdot e^{-kx}$ (note the sign), we have $-k = \ln(a)$, and may therefore calculate the half-life as[4]

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{-k} = \frac{\ln(2^{-1})}{-k} = \frac{-1 \cdot \ln(2)}{-k} = \frac{\ln(2)}{k} .$$

[4] $\frac{1}{2} = 2^{-1}$ follows from the exponentiation rule $a^{-n} = \frac{1}{a^n}$.

Thus, we may calculate the half-life using the formula

$$T_{\frac{1}{2}} = \frac{\ln(2)}{k} \qquad (\text{when } f(x) = b \cdot e^{-kx}) .$$

4.3 Exercises

Exercise 4.1
Calculate

a) $\log_{10}(1000)$

b) $\log_3(9)$

c) $\log_4(64)$

d) $\log_{10}(0,1)$

e) $\log_7(1)$

f) $\log_2(16)$

g) $\log_2\left(\frac{1}{8}\right)$

h) $\log_5(625)$

Exercise 4.2
Calculate

a) $\log_3(3) + \log_3(9)$

b) $\log_{10}(1000) - \log_{10}(100)$

c) $\log_9(9^8)$

d) $\log_6(\sqrt{6})$

Exercise 4.3
Calculate the following numbers by rewriting the calculation to a different base logarithm.

a) $\log_3(8)$
b) $\log_8(139)$
c) $\log_{12}(45)$
d) $\log_5(0.6)$
e) $\log_6(3987)$
f) $\log_{73}(932\,108)$

Exercise 4.4
Without using a calculator, determine

a) $\log(2) + \log(5)$
b) $2 \cdot \log(5) + \log(4)$
c) $\log(25) + 2 \cdot \log(2)$
d) $\log(25) - 2 \cdot \log(5)$
e) $3 \cdot \log(2) + 3 \cdot \log(5)$
f) $\log(8) + 3 \cdot \log(5)$
g) $\log(8) - 3 \cdot \log(2)$
h) $\log(8) - \log(2) + \log\left(\frac{1}{4}\right)$

Exercise 4.5
Without using a calculator, solve the following equations:

a) $\log(x) = 4$
b) $\log(x) = -1$
c) $\log(x) = 3$
d) $\log(x) = 2$
e) $\log(x) = -2$
f) $\log(x) = 0$

Exercise 4.6
Solve the following equations:

a) $\log(x - 3) = 4$
b) $\log(4x^2) = 3$
c) $\log(2x - 6) = 2$
d) $\log(10x + 25) = 3$
e) $\log(x + 5) = 2 - \log(x - 5)$
f) $\log(x^2 + 4x + 4) = 3$

Exercise 4.7
Use the rules for logarithms to rewrite these expressions as much as possible:

a) $\log(100x^3)$
b) $\log(1000t)$
c) $\log(10p^4 \cdot q^2)$
d) $\log\left(\sqrt{1000u^3}\right)$

Exercise 4.8
The two numbers a and b satisfy $\ln(a) = 1.5$ and $\ln(b) = 0.5$.

Use this to calculate (without using a calculator):

a) $\ln(a \cdot b)$
b) $\ln\left(\frac{a}{b}\right)$
c) $\ln(a^4)$
d) $\ln\left(\frac{b}{a}\right)$
e) $\ln(b^7)$
f) $\ln(a^3 \cdot b^{10})$

Exercise 4.9
Solve the following equations:

a) $2^x = 50$
b) $\ln(x) = 3.7$
c) $\ln(4x - 3) = 5.1$
d) $4 \cdot 6^{x+2} = 100$
e) $6 \cdot 3.9^x = 78$
f) $5.2 \cdot 7^{3-x} = 81.5$

Exercise 4.10
In each case below, rewrite the exponential function to the form $f(x) = b \cdot e^{kx}$ and calculate its doubling time/half-life.

a) $f(x) = 0.45 \cdot 7.8^x$
b) $f(x) = 1.7 \cdot 0.56^x$
c) $f(x) = 45.6 \cdot 1.2^x$
d) $f(x) = 6.1 \cdot 0.34^x$

Exercise 4.11
Determine a formula for the exponential function, whose graph passes through the two points P and Q. The formula must be of the form $f(x) = b \cdot e^{kx}$.

a) $P(2, 4)$ og $Q(6, 7)$
b) $P(1, 1)$ og $Q(4, 26)$
c) $P(-3, 7)$ og $Q(2, 1)$
d) $P(0, 17)$ og $Q(5, 1)$

5 Power functions

Definition 5.1

A power function is a function of the form

$$f(x) = b \cdot x^a, \qquad x > 0\,,$$

where a and b are two constants, and $b > 0$.

Note that power functions are only defined for positive values of x. This is because values of a exist where x^a is not defined for every possible value of x.[1]

[1] An example is x^{-1} which is equal to $\frac{1}{x}$. Because we cannot divide by 0, this function is undefined for $x = 0$.

Because we also demand that $b > 0$, the graph of a power function will lie entirely in the first quadrant, i.e. the x- and y-coordinates of points on the graph of the function are always positive.

Since a power function is not defined for $x = 0$, it makes no sense to talk about a y-axis intercept. However, the definition implies that the graph of a power function $f(x) = b \cdot x^a$ always passes through the point $(1, b)$, because

$$f(1) = b \cdot 1^a = b\,.$$

Here, we present a few examples of power functions:

Example 5.2 The area A of a circle with radius r is

$$A = \pi \cdot r^2\,.$$

Here, the area A is a power function of the radius r. The two constants a and b are $a = 2$ and $b = \pi$.

Example 5.3 The speed v of a tsunami wave (in km/h) is a power function of the sea depth d (in metres),[1]

$$v = 11.2 \cdot d^{0.5}\,.$$

Here, $a = 0.5$ and $b = 11.2$.

5.1 The graph of a power function

The number a in definition 5.1 is called the *exponent*. This number determines the shape of the graph of a power function. Figure 5.1 shows how the shape of the graph changes for different values of a.

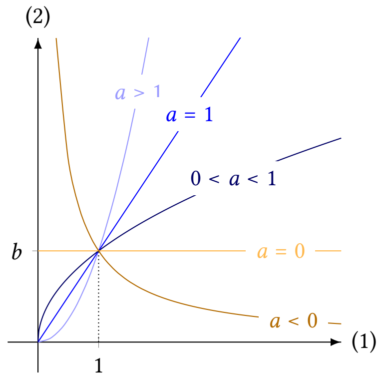


Figure 5.1: The graphs of power functions may have quite different shapes.

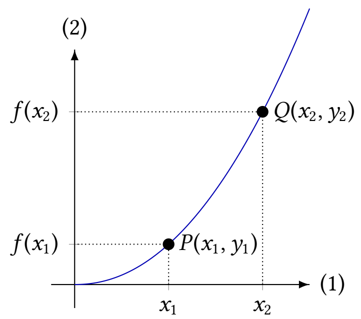


Figure 5.2: Two points on the graph of a power function.

We have the following theorem:

Theorem 5.4

For a power function $f(x) = b \cdot x^a$, we have:

1. If $a > 0$, the function is increasing.
2. If $a < 0$, the function is decreasing.

If we know the graph of a power function, we may use two points on the graph to calculate the two numbers a and b. (See figure 5.2.)

Theorem 5.5

If the graph of a power function $f(x) = b \cdot a^x$ passes through the points (x_1, y_1) and (x_2, y_2), then

$$a = \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{x_2}{x_1}\right)} \qquad \text{and} \qquad b = \frac{y_1}{x_1^a} .$$

Proof

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the graph of $f(x) = b \cdot x^a$, then

$$\begin{aligned} y_2 &= b \cdot x_2^a \\ y_1 &= b \cdot x_1^a \end{aligned} \tag{5.1}$$

We divide these two equations, and get

$$\begin{aligned} \frac{y_2}{y_1} &= \frac{b x_2^a}{b x_1^a} && \Longleftrightarrow \\ \frac{y_2}{y_1} &= \frac{x_2^a}{x_1^a} && \Longleftrightarrow \\ \frac{y_2}{y_1} &= \left(\frac{x_2}{x_1}\right)^a . \end{aligned}$$

This is an exponential equation, so we need to use logarithms to solve it. We then get

$$\begin{aligned} \log\left(\frac{y_2}{y_1}\right) &= \log\left(\left(\frac{x_2}{x_1}\right)^a\right) && \Longleftrightarrow \\ \log\left(\frac{y_2}{y_1}\right) &= a \cdot \log\left(\frac{x_2}{x_1}\right) && \Longleftrightarrow \\ \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{x_2}{x_1}\right)} &= a . \end{aligned}$$

This proves the formula for a.

To prove the formula for b, we look again at equation (5.1):

$$y_1 = b \cdot x_1^a \qquad \Longleftrightarrow \qquad \frac{y_1}{x_1^a} = b .$$

Thus, the formula for b is also proven. ∎

5.2 Power growth

A power function increases in such a way that if the indendent variable is multiplied by some fixed number, the dependent variable is also multiplied by a fixed number.[2] We have the following theorem:

[2] A different fixed number.

Theorem 5.6

If $f(x) = b \cdot x^a$ is a power function, then when x is multiplied by a number k, the function value $f(x)$ is multplied by k^a.

Proof
If x is multiplied by k, the new function value is $f(k \cdot x)$. But

$$f(k \cdot x) = b \cdot (k \cdot x)^a = b \cdot k^a \cdot x^a = k^a \cdot b \cdot x^a = k^a \cdot f(x) \,.$$

So, the function value is multiplied by k^a. ■

Example 5.7 Table 5.3 shows how the function $f(x) = 4x^3$ increases. In the formula for this function, the exponent is $a = 3$. Therefore, whenever x is multiplied by 2, y is multiplied by 2^3, i.e. 8.

Table 5.3: Growth of $f(x) = 4x^3$.

x	y
1	4
2	32
4	256
8	2048

Example 5.8 A power function $f(x) = b \cdot x^2$ has a graph passing through the point $(3, 7)$. Here, we do not know the value of b, but we may still find a second point on the graph.

If we multiply x by 4,[3] we get the new value $3 \cdot 4 = 12$. We can then find the new function value by multiplying the old value (7) by 4^2 (because the exponent is $a = 2$). We then get

[3] There is nothing special about the number 4, it could have been any positive number.

$$7 \cdot 4^2 = 7 \cdot 16 = 112 \,.$$

So, the graph of this function also passes through the point $(12, 112)$.

Multiplying by a number corresponds to relative growth. If we multiply by some number k, we have a relative growth of $k - 1$. It would make sense, therefore, to write the number k in theorem 5.6 as $1 + r_x$, where r_x is the relative growth of the independent variable x. The number r_y is then the relative growth of the dependent variable y, and we get

$$k = 1 + r_x \qquad \text{og} \qquad k^a = 1 + r_y \,.$$

We write this as a theorem:

Theorem 5.9

If the x-value of a power function $f(x) = b \cdot x^a$ has a relative growth of r_x, the function value has a relative growth of r_y, and

$$1 + r_y = (1 + r_x)^a \,.$$

Example 5.10 The function $f(x) = 4.2 \cdot x^{0.5}$ is an increasing function. If x increases by 80%, we have $r_x = 0.80$. I.e.

$$1 + r_y = (1 + 0.80)^{0.5} = 1.342 \ .$$

r_y must then equal 0.342, which corresponds to 34.2%. Therefore, each time x increases by 80%, y increases by 34.2%.

Example 5.11 The function $f(x) = 5x^{-2}$ is a decreasing function with $a = -2$. If x increases by 40%, $r_x = 0.40$, i.e.

$$1 + r_y = (1 + 0.40)^{-2} = 0.510 \ .$$

This corresponds to

$$r_y = 0.510 - 1 = -0.490 = -49\% \ .$$

So, if x increases by 40%, y *decreases* by 49%.

Example 5.12 If we have a function $f(x) = 2x^3$, and we know that y has increased by 50%, how much did x then increase?

We solve this problem by setting $r_y = 0.50$ in the formula, where we get

$$1 + 0.50 = (1 + r_x)^3 \ .$$

We then solve this equation,

$$\begin{aligned} 1 + 0.50 &= (1 + r_x)^3 && \iff \\ \sqrt[3]{1.50} &= 1 + r_x && \iff \\ \sqrt[3]{1.50} - 1 &= r_x && \iff \\ 0.145 &= r_x \ . \end{aligned}$$

Therefore, x increased by 14.5% if y increased by 50%.

5.3 Proportionality

Two variables y and x are said to be directly proportional when

$$y = k \cdot x \ ,$$

where k is a constant. If we instead denote the constant by b, we get the relationship

$$y = b \cdot x = b \cdot x^1 \ ,$$

which means that y is a power function of x with $a = 1$.

In the same way, inverse proportionality is also a power function. Two variables x and y are inversely proportional when $x \cdot y = k$, which we may also write as[4]

$$y = \frac{b}{x} = b \cdot \frac{1}{x} = b \cdot x^{-1} \ .$$

So, if y and x are inversely proportional to x, y is a power function of x with $a = -1$.

We then have:

[4] We use the identity $\frac{1}{x^n} = x^{-n}$ to rewrite the formula.

Theorem 5.13

For a power function $y = b \cdot x^a$, we have:

1. If $a = 1$, y and x are directly proportional.
2. If $a = -1$, y and x are inversely proportional.

Therefore, a direct proportionality may be described by the power function

$$f(x) = b \cdot x \ .$$

The graph of this function is a straight line through $(0, 0)$,[5] i.e. this is in fact also a linear function with slope b, intercepting the y-axis at 0.

[5]Strictly speaking $x > 0$ when f is a power function, but in this case nothing prevents us from letting x assume negative values.

So, direct proportionality may be viewed as a power function with exponent 1, but also as a linear function intercepting the y-axis at 0.

5.4 Exercises

Exercise 5.1
A power function $f(x)$ has a graph passing through the points $(1, 3)$ og $(4, 48)$.

a) Determine a formula for the function.

b) Solve the equation $f(x) = 12$.

Exercise 5.2
The graph of a power function f passes through the points $(4, 12)$ og $(16, 48)$.

a) Determine a formula for this function.

b) What is the relative function growth when x has a relative growth of 0.25?

Exercise 5.3
A power function is given by $f(x) = 3x^{2.4}$.

How many percent does the function value increase when x increases by 7%?

Exercise 5.4
A power function is given by $f(x) = 5 \cdot x^2$.

a) If x is doubled, how many times larger will the function value be?

b) If the function value increases to 9 times its initial value, how many times larger is x?

Exercise 5.5
"If the price of a ticket is increased by 10% the number of passengers will decrease by 3%."

a) Write down a mathematical model of this statement.

b) How many percent will the number of passengers decrease if the price of a ticket is increased by 15%.

c) How many percent must the price of a ticket be decreased if we want the number of passengers to increase by 25%?

6 Polynomials

The function

$$f(x) = 3x^2 + x - 4$$

belongs to the group of *polynomials*, which are a type of function used in many branches of mathematics. This is because, as it turns out, polynomials have certain nice properties.

In general, we define a *polynomial* in this way:

Definition 6.1

A *polynomial* is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \ ,$$

where the coefficients $a_0, \ldots, a_n$ are real numbers, and $a_n \neq 0$.

The number n which must be a positive integer, is called the *degree* of the polynomial.

Some examples of polynomials are:

$$\begin{aligned}
&\textit{First degree} \text{ polynomial:} && f(x) = 3x + 1 \\
&\textit{Second degree} \text{ polynomial:} && g(x) = 4x^2 - 3x + 5 \\
&\textit{Third degree} \text{ polynomial:} && h(x) = x^3 + 7x - 13 \\
&\textit{Fourth degree} \text{ polynomial:} && m(x) = 8x^4 + 7x^2 \\
&\textit{Seventeenth degree} \text{ polynomial:} && p(x) = x^{17} + 4x^9 \ .
\end{aligned}$$

When we write the formula of a polynomial, we usually sort the terms, so that the exponents are ordered and the largest is written first. This is not stricly necessary, but it makes the task of determining the degree a lot easier since the degree corresponds to the largest exponent.[1]

[1] A special case is the polynomials of degree 0, which are the constant functions; e.g. $f(x) = 9$ or $g(x) = -14$.

Example 6.2 The polynomial $f(x) = x + 4 - 3x^2$ may also be written as

$$f(x) = -3x^2 + x + 4 \ ,$$

and here it is easy to see that f is a second degree polynomial.

First degree polynomials are actually linear functions, so we will skip them in this chapter. Most of the chapter will instead deal with the properties of second degree polynomials—with a concluding section on polynomials of higher degree.

6.1 Second degree polynomials

A *second degree polynomial* is a polynomial of degree 2. According to definition 6.1, this is a function of the form

$$f(x) = ax^2 + bx + c\ , \tag{6.1}$$

where a, b and c are three numbers, and $a \neq 0$.[2]

[2] In the definition, the coefficients are denoted by a_2, a_1 and a_0, but because a second degree polynomial only has three coefficients, it is easier to denote them by a, b and c.

The simplest second degree polynomial imaginable is one where the coefficients b and c are both 0, i.e.

$$p(x) = ax^2\ .$$

The graph of $p(x) = ax^2$ is shown in figure 6.1. A graph of this form is called a *parabola*. As the figure shows, the shape of the graph depends on the value of the coefficient a: If $a > 0$, the parabola opens upwards; if $a < 0$, it opens downwards.

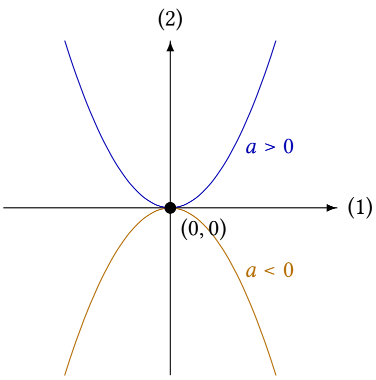


Figure 6.1: The graph of $p(x) = ax^2$ when $a > 0$ and $a < 0$.

The reason is that x^2 is always a positive number. The sign of the function value, therefore, only depends on the sign of a.

The figure also shows that the y-axis is a symmetry axis of this parabola. The reason is that $(-x)^2 = x^2$, i.e. the polynomial $p(x)$ has the same function values for x and $-x$.

Lastly, we also see from the figure that no matter what the value of a might be, the parabola "changes direction" at the point $(0, 0)$. We call this point the *vertex* of the parabola.

If we want to draw a parabola with a vertex in (x_0, y_0), we can shift the graph of $p(x)$. This is shown in figure 6.2. The new parabola will then have the line $x = x_0$ as its axis of symmetry.

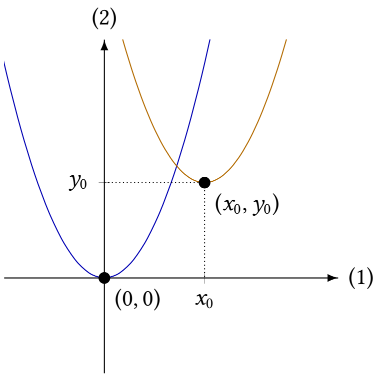


Figure 6.2: The graph of $p(x) = ax^2$ shifted to the graph of $f(x) = a(x - x_0)^2 + y_0$.

Using theorem 1.13 we may derive the following:

Theorem 6.3

The parabola with (x_0, y_0) as its vertex is the graph of the function

$$f(x) = a(x - x_0)^2 + y_0\ .$$

Proof

We obtain the parabola with vertex at (x_0, y_0) by shifting the graph of the parabola with vertex at $(0, 0)$ by (x_0, y_0).

The parabola with vertex at $(0, 0)$ is the graph of the function $p(x) = ax^2$. According to theorem 1.13, the parabola with vertex at (x_0, y_0), therefore, is the graph of

$$f(x) = p(x - x_0) + y_0 = a(x - x_0)^2 + y_0\ . \qquad \blacksquare$$

The formula of the function in theorem 6.3 does not look like the second degree polynomial in (6.1). But it turns out that it is possible to rewrite one form into the other.

Example 6.4 The graph of the function $f(x) = 3 \cdot (x-2)^2 - 7$ is a parabola with vertex at $(2, -7)$.

If we square the parenthesis and reduce, we may rewrite the formula and get

$$\begin{aligned} f(x) &= 3(x-2)^2 - 7 \\ &= 3(x^2 + (-2)^2 - 2 \cdot 2 \cdot x) - 7 \\ &= 3(x^2 + 4 - 4x) - 7 \\ &= 3x^2 + 12 - 12x - 7 \\ &= 3x^2 - 12x + 5 \ . \end{aligned}$$

So, the formula $f(x) = 3(x-2)^2 - 7$ may also be written as

$$f(x) = 3x^2 - 12x + 5 \ ,$$

which corresponds to (6.1) with the coefficients

$$a = 3, \quad b = -12 \quad \text{and} \quad c = 5 \ .$$

We can generalise the calculations in example 6.4 if we look at the second degree polynomial $f(x) = a(x - x_0)^2 + y_0$. We then get

$$\begin{aligned} f(x) &= a(x - x_0)^2 + y_0 \\ &= a(x^2 + x_0^2 - 2x_0 x) + y_0 \\ &= ax^2 + ax_0^2 - 2ax_0 x + y_0 \\ &= ax^2 + (-2ax_0)x + (ax_0^2 + y_0) \ . \end{aligned}$$

If this has to correspond to the formula

$$f(x) = ax^2 + bx + c \ ,$$

the coefficients must be equal. This implies that

$$b = -2ax_0 \quad \text{og} \quad c = ax_0^2 + y_0 \ . \tag{6.2}$$

The equations (6.2) can be used to calculate the coefficients b and c when we know the vertex (x_0, y_0). Usually, we have the second degree polynomial in the form (6.1), and we would therefore like to be able to calculate the vertex when we know the three coefficients a, b and c.

We want the formula for the vertex to be simple, so we introduce the *discriminant*,[3]

$$d = b^2 - 4ac \ .$$

[3] The discriminant is used to calculate more than just the vertex, so it makes sense to define this quantity. It reappears in section 6.2 below.

We then have this theorem:

Theorem 6.5

The vertex of the second degree polynomial $f(x) = ax^2 + bx + c$ is at (x_0, y_0), where

$$x_0 = -\frac{b}{2a} \quad \text{and} \quad y_0 = -\frac{d}{4a} \ .$$

$d = b^2 - 4ac$ is the *discriminant*.

Proof
To prove the theorem, we look again at the equation (6.2). Here, we see that

$$b = -2ax_0 \qquad \Longleftrightarrow \qquad -\frac{b}{2a} = x_0 \ .$$

This proves the formula for x_0.

Because $c = ax_0^2 + y_0$, we get

$$y_0 = c - ax_0^2 \ .$$

We have just shown that $x_0 = -\frac{b}{2a}$, so

$$\begin{aligned} y_0 &= c - a\left(-\frac{b}{2a}\right)^2 = c - a \cdot \frac{b^2}{4a^2} = c - \frac{b^2}{4a} \\ &= \frac{4ac}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} = -\frac{b^2 - 4ac}{4a} = -\frac{d}{4a} \ . \end{aligned}$$

Thus, we have also proven the formula for y_0. ∎

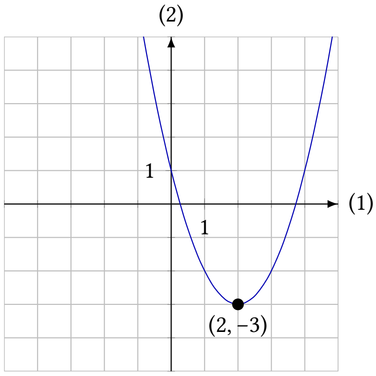


Figure 6.3: The graph $f(x) = x^2 - 4x + 1$ has its vertex at $(2, -3)$.

Example 6.6 The graph of the second degree polynomial

$$f(x) = x^2 - 4x + 1$$

is shown in figure 6.3. If we want to determine the vertex of this parabola, we first find the coefficients of the polynomial. They are

$$a = 1 \ , \quad b = -4 \quad \text{and} \quad c = 1 \ .$$

We can now calculate the x-coordinate of the vertex:

$$x_0 = -\frac{b}{2a} = -\frac{-4}{2 \cdot 1} = 2 \ .$$

To calculate the y-coordinate, we first calculate the discriminant:

$$d = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 1 = 12 \ .$$

Then, the y-coordinate of the vertex is

$$y_0 = -\frac{d}{4a} = -\frac{12}{4 \cdot 1} = -3 \ .$$

So, the parabola has its vertex at $(2, -3)$, which we also see in the figure.

6.2 Quadratic equations

A parabola may be placed, so that it intercepts the x-axis. The values of x where the parabola intercepts the x-axis are called the *roots* of the polynomial.

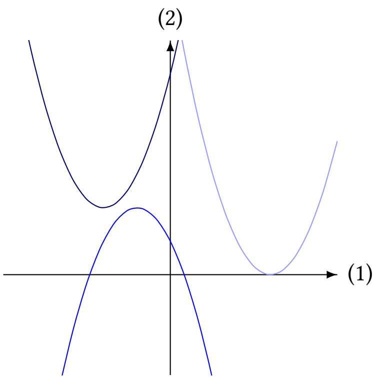


Figure 6.4: A parabola may have 2, 1 or no roots.

Second degree polynomials may have 2, 1 or no roots, depending on how the parabola is placed in the coordinate system. This is illustrated in figure 6.4, where one of the parabolas intercepts the x-axis (2 roots), another parabola just touches the x-axis (1 root), and the last parabola has no point in common with the x-axis (no roots).

As noted above, we find the roots where the parabola intercepts the x-axis. On the x-axis, $y = 0$, i.e. the roots are found where $f(x) = 0$. This means that we can find the roots by solving the *quadratic equation*

$$ax^2 + bx + c = 0 \ .$$

In this equation, it is not easy to see straight away, how we might isolate x; however, it is possible to derive a solution formula for such an equation.

Theorem 6.7

To solve the quadratic equation

$$ax^2 + bx + c = 0 \ ,$$

we first calculate the discriminant $d = b^2 - 4ac$.

We then have:

1. If $d < 0$, the equation has no solutions.
2. If $d \geq 0$, the equation has the solutions $x = \dfrac{-b \pm \sqrt{d}}{2a}$.

Proof

Combining theorem 6.3 and theorem 6.5, we find that the second degree polynomial

$$f(x) = ax^2 + bx + c$$

may be written as[4]

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(-\frac{d}{4a}\right) = a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{d}{4a} \ .$$

[4]We arrive at this expression by inserting the formulas $x_0 = -\frac{b}{2a}$ and $y_0 = -\frac{d}{4a}$ into the formula $f(x) = a(x - x_0)^2 + y_0$.

This means that the equation $ax^2 + bx + c = 0$ may be written as

$$a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{d}{4a} = 0$$

If $d \geq 0$, this equation may be rewritten[5] to

$$\begin{aligned} a \cdot \left(x + \frac{b}{2a}\right)^2 &= \frac{d}{4a} && \iff \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{d}{4a^2} && \iff \\ \left(x + \frac{b}{2a}\right)^2 &= \left(\frac{\sqrt{d}}{2a}\right)^2 \ . \end{aligned}$$

[5]If $d < 0$, this equation cannot be solved. Thus, the sign of the discriminant determines whether the equation has any solutions or not.

This equation is satisfied if[6]

[6]There are two possible solutions because $(-h)^2 = h^2$, which means that both $-h$ and h are solutions of the equation $x^2 = h^2$.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{d}}{2a} .$$

We then isolate x and get

$$\begin{aligned} x + \frac{b}{2a} &= \pm\frac{\sqrt{d}}{2a} && \Longleftrightarrow \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{d}}{2a} && \Longleftrightarrow \\ x &= \frac{-b \pm \sqrt{d}}{2a} , \end{aligned}$$

which proves the formula. ■

If we look at the formula in theorem 6.7, we notice that if $d = 0$, we only have a single solution because[7]

$$\frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a} .$$

[7] So, if the discriminant is 0, the solution to the equation is $x = -\frac{b}{2a}$.

In this case, we say that the polynomial has a *double root*. This corresponds to the situation where the parabola just touches the x-axis at a single point (see figure 6.4).

Next, we present a few examples of how to use the formula.

Example 6.8 To find the roots of the second degree polynomial

$$f(x) = 2x^2 + 2x - 12 ,$$

we first find the coefficients of the polynomial. They are

$$a = 2, \quad b = 2 \quad \text{and} \quad c = -12 .$$

Next, we calculate the discriminant

$$d = b^2 - 4ac = 2^2 - 4 \cdot 2 \cdot (-12) = 100 .$$

d is positive, so we have to roots. We calculate these using the formula:

$$x = \frac{-b \pm \sqrt{d}}{2a} = \frac{-2 \pm \sqrt{100}}{2 \cdot 2} = \frac{-2 \pm 10}{4} .$$

The two roots are

$$x = \frac{-2 - 10}{4} = -3 \quad \text{and} \quad x = \frac{-2 + 10}{4} = 2 ,$$

which is also illustrated in figure 6.5.

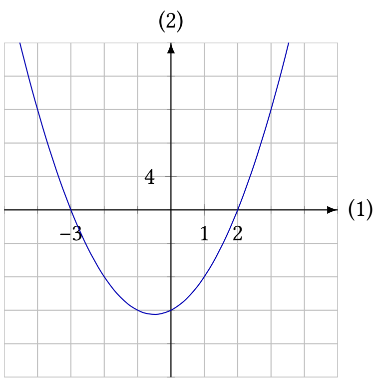


Figure 6.5: The polynomial $f(x) = 2x^2 + 2x - 12$ has the roots -3 and 2.

Example 6.9 Here, we solve the equation

$$-x^2 + 8x - 16 = 0 .$$

The coefficients are

$$a = -1\ , \quad b = 8 \quad \text{and} \quad c = -16\ ,$$

and the discriminant is

$$d = b^2 - 4ac = 8^2 - 4 \cdot (-1) \cdot (-16) = 0\ .$$

Therefore, the equation has the solution

$$x = -\frac{b}{2a} = -\frac{8}{2 \cdot (-1)} = 4\ .$$

Example 6.10 To find the roots of the polynomial

$$f(x) = 3x^2 + 2x + 5\ ,$$

we calculate the discriminant

$$d = b^2 - 4ac = 2^2 - 4 \cdot 3 \cdot 5 = 4 - 60 = -56\ .$$

Because the discriminant is negative, the polynomial has no roots. This is illustrated in figure 6.6.

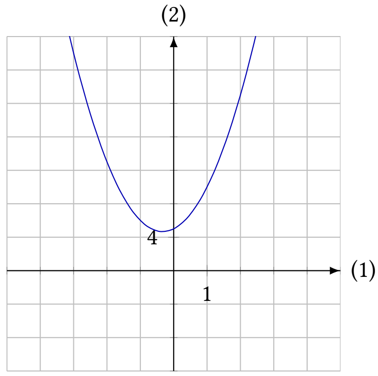


Figure 6.6: The polynomial $f(x) = 3x^2 + 2x + 1$ has no roots.

Simple quadratic equations

It turns out that some quadratic equations can be solved without the solution formula. If either b or c is equal to 0, there is an easier way to solve these equations.

Example 6.11 (b = 0) In the quadratic equation

$$3x^2 - 75 = 0\ ,$$

the coefficient $b = 0$. We can then solve the equation like this:

$$\begin{aligned} 3x^2 - 75 &= 0 && \iff \\ 3x^2 &= 75 && \iff \\ x^2 &= 25 && \iff \\ x &= \pm\sqrt{25} && \iff \\ x &= \pm 5\ . \end{aligned}$$

Here, we only need to remember that there are two possible solutions when we take the square root: a positive and a negative one.

Example 6.12 (c = 0) In the quadratic equation

$$2x^2 + 14x = 0\ ,$$

$c = 0$. Here, we can solve the equation by factorisation and use of the zero product rule:

$$\begin{aligned} 2x^2 + 14x &= 0 && \iff \\ 2x \cdot x + 7 \cdot 2x &= 0 && \iff \\ 2x \cdot (x + 7) &= 0 && \iff \\ 2x = 0 \quad &\vee \quad x + 7 = 0 && \iff \\ x = 0 \quad &\vee \quad \quad x = -7\ . \end{aligned}$$

6.3 Interpreting the coefficients

There is a direct connection between the graph of a second degree polynomial $f(x) = ax^2 + bx + c$ and the coefficients a, b and c, and the discriminant d. In this section, we take a closer look at this connection.

In section 6.1 above, we demonstrated that the sign of a determines whether the parabola opens upwards or downwards.[8]

[8]The argument concerned the polynomial $p(x) = ax^2$, but $f(x) = ax^2 + bx + c$ is a shift of $p(x)$, so the same argument applies here.

The x-coordinate of the vertex is $x_0 = -\frac{b}{2a}$. Therefore, if a and b have the same sign, x_0 is negative. In this case, the vertex lies to the left of the y-axis. For the same reason, the vertex must lie to the right of the y-axis when a and b have different signs.

If $b = 0$, we get $x_0 = -\frac{0}{2a} = 0$; in this case, the vertex lies on the y-axis.

If we let $x = 0$ in the formula for a second degree polynomial, we get

$$f(0) = a \cdot 0^2 + b \cdot 0 + c\ ,$$

i.e. the parabola passes through the point $(0, c)$. So, c is the y-axis intercept.

The connection between the discriminant d and the parabola was discussed in the previous section. Here, we showed that when $d < 0$, the parabola does not intercept the x-axis.

All of these arguments lead to the following theorem:

Theorem 6.13

Let a second degree polynomial be given by

$$f(x) = ax^2 + bx + c\ ,$$

and let d be the discriminant.

Then, the graph of the polynomial is a parabola, and the following holds:

1. If $a > 0$, the parabola opens upwards; if $a < 0$, the parabola opens downwards.
2. If a and b have the same sign, the vertex lies to the left of the y-axis. If they have opposite signs, the vertex lies to the right of the y-axis; and if $b = 0$, the vertex lies on the y-axis.
3. The parabola intercepts the y-axis at $(0, c)$.
4. If $d > 0$, the parabola intercepts the x-axis twice; if $d < 0$, the parabola does not intercept the x-axis; and if $d = 0$, the parabola touches the x-axis at one point.

6.4 Factoring polynomials

If a second degree polynomial $f(x) = ax^2 + bx + c$ has two roots, these are given by

$$r_1 = \frac{-b + \sqrt{d}}{2a} \qquad \wedge \qquad r_2 = \frac{-b - \sqrt{d}}{2a} \; .$$

In this case, we may also write the polynomial as

$$f(x) = a(x - r_1)(x - r_2) \; .$$

We say that we have *factored* the polynomial.[9]

Example 6.14 If we want to factor the second degree polynomial $f(x) = 3x^2 - 3x - 6$, we must first find its roots. To do this, we calculate the discriminant,

$$d = b^2 - 4ac = (-3)^2 - 4 \cdot 3 \cdot (-6) = 81 \; .$$

The two roots are therefore

$$r_1 = \frac{-b + \sqrt{d}}{2a} = \frac{-(-3) + \sqrt{81}}{2 \cdot 3} = 2$$

and

$$r_2 = \frac{-b - \sqrt{d}}{2a} = \frac{-(-3) - \sqrt{81}}{2 \cdot 3} = -1 \; .$$

We can now factor the polynomial:

$$\begin{aligned} f(x) &= a(x - r_1)(x - r_2) \\ &= 3(x - 2)(x - (-1)) \\ &= 3(x - 2)(x + 1) \; . \end{aligned}$$

When a polynomial is factored, we can see the roots straight away.[10] If a polynomial has no roots, it cannot be factored.

The two second degree polynomials $f(x) = ax^2 + bx + c$ and $p(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ must have the same roots.[11] If we factor $p(x)$, we then get

$$p(x) = (x - r_1)(x - r_2) \; ,$$

because the coefficient of the second degree term is 1.

If we multiply these parentheses we get

$$p(x) = x^2 - r_1 x - r_2 x + r_1 r_2 = x^2 - (r_1 + r_2)x + r_1 r_2 \; .$$

This polynomial must be equal to the original polynomial, i.e. the coefficients must be equal. Therefore,

$$-\frac{b}{a} = r_1 + r_2 \qquad \wedge \qquad \frac{c}{a} = r_1 r_2 \; .$$

Because the polynomial $f(x) = ax^2 + bx + c$ has the same roots, this must also be true for the polynomial f.

We can use this to form educated guesses about the roots of a second degree polynomial.

[9] If we want to prove that this is correct, we can reduce

$$a\left(x - \frac{-b + \sqrt{d}}{2a}\right)\left(x - \frac{-b - \sqrt{d}}{2a}\right)$$

and see, whether this yields $ax^2 + bx + c$.

[10] If a second degree polynomial has one root r, the factorisation will be $f(x) = a(x - r)^2$, and r is called a *double root*.

[11] The reason is that whenever $ax^2 + bx + c = 0$, we must also have $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Example 6.15 If we want to guess the roots of the polynomial $f(x) = 4x^2 - 12x + 8$, we begin by calculating

$$-\frac{b}{a} = -\frac{-12}{4} = 3 ,$$

and

$$\frac{c}{a} = \frac{8}{4} = 2 .$$

From the previous argument, we know that the sum of the roots $(r_1 + r_2)$ must be equal to 3, while their product $(r_1 r_2)$ must be equal to 2. This is only true for the numbers 1 and 2, and therefore these are the roots.

We sum up our results in the following theorem:

Theorem 6.16

If a second degree polynomial $f(x) = ax^2 + bx + c$ has at least one root, we can factor the polynomial, i.e. write it in the form

$$f(x) = a(x - r_1)(x - r_2) ,$$

where r_1 and r_2 are the two roots (if there is only one root, then $r_1 = r_2$).

The two roots also satisfy the equations

$$r_1 + r_2 = -\frac{b}{a} ,$$

and

$$r_1 r_2 = \frac{c}{a} .$$

6.5 Polynomials of higher degree

Polynomials of a degree higher than 2 are not as simple to describe. Figure 6.7 shows graphs of polynomials of degree 3 to 6. As the figure shows, the graphs “turn” more often, the higher the degree of the polynomial. The points where the polynomials turn are known as *turning points*.[12]

Because the graphs may have many turning points, it is possible for the graphs to intercept the x-axis many times. We cannot, therefore, find simple solution formulas for the roots.[13]

In the previous section, we demonstrated how to factor second degree polynomials. It is also possible to factor polynomials of a higher degree if we know the roots. In general, if r is a root in the polynomial $p(x)$, we may write $p(x)$ as

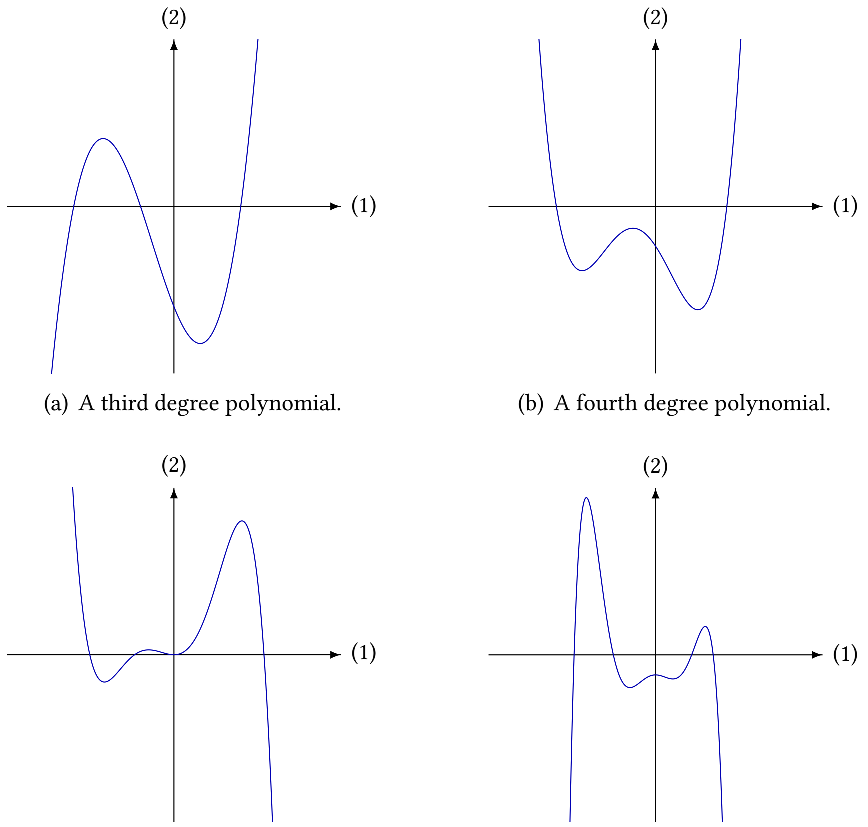
$$p(x) = (x - r) \cdot q(x) ,$$

where $q(x)$ is also a polynomial. The degree of q is 1 less than the degree of p. So, if p is a fourth degree polynomial, q is a third degree polynomial, etc.

This implies that a polynomial of degree n can have at most n roots. We have the following theorem:

[12] *Turning points* then becomes a collective term for maxima and minima.

[13] Solution formulas exist for the roots of third and fourth degree polynomials, but they are anything but simple. For polynomials of degree 5 or higher, it has been proven that it is impossible to find a general solution formula.[3]



(a) A third degree polynomial.

(b) A fourth degree polynomial.

(c) A fifth degree polynomial.

(d) A sixth degree polynomial.

Figure 6.7: Here, we see the graphs of four polynomials of a degree higher than 2. As the figure shows, the graphs "turn" more, the higher the degree.

Theorem 6.17

1. A polynomial of degree n has at most n roots.
2. An equation of degree n has at most n solutions.

Because no general solution formulas for the roots of a polynomial of degree n exist, we need to use different approaches. CAS's usually have a built-in "factor" function, which can be used to factor polynomials.

Example 6.18 We can rewrite the polynomial $f(x) = 2x^3 - 16x^2 + 2x + 84$ using a CAS, and get

$$f(x) = 2 \cdot (x + 2) \cdot (x - 3) \cdot (x - 7) ,$$

which immediately shows us that this third degree polynomial has the roots -2, 3 and 7.

Example 6.19 We can use a CAS to factor the fourth degree polynomial $f(x) = x^4 - x^3 - 19x^2 - x - 20$. We get

$$f(x) = (x - 5) \cdot (x + 4) \cdot (x^2 + 1) .$$

So, the polynomial has the two roots -4 and 5.

We cannot factor this polynomial any further because the second degree polynomial $x^2 + 1$ has no roots. This is also the reason why the polynomial has only two roots, even though $f(x)$ is a fourth degree polynomial.

In the example above, we looked at a fourth degree polynomial which had only 2 roots. It is actually possible to construct a fourth degree polynomial which has no roots at all. This is, however, impossible for a fifth degree polynomial. Figure 6.7 hints at the reason. We infer from this figure that a polynomial of odd degree always has at least one root.

We can use factorisation to construct a polynomial with specific roots. This concluding example shows how:

Example 6.20 We construct a third degree polynomial with the roots −1, 4 and 7 by writing

$$f(x) = (x - (-1)) \cdot (x - 4) \cdot (x - 7) \ .$$

If we multiply the parentheses, we get

$$f(x) = x^3 - 10x^2 + 17x + 28 \ .$$

6.6 Exercises

Exercise 6.1
Which of these functions are polynomials? And what is their degree?

a) $f_1(x) = x^2 + 6x - 2$

b) $f_2(x) = x + 5x^8 - 1 + x^2$

c) $f_3(x) = x^{-3} + 2x^6 - 6x$

d) $f_4(x) = 7$

e) $f_5(x) = 2^x - x^2$

f) $f_6(x) = 213x^{89\,364\,521}$

g) $f_7(x) = x^{\frac{3}{2}} + 4x^{\frac{7}{10}}$

Exercise 6.2
Find the vertices (x_0, y_0) of the graphs of the second degree polynomials below. Then, rewrite the formulas to the form $f(x) = ax^2 + bx + c$.

a) $f_1(x) = (x - 4)^2 + 3$

b) $f_2(x) = -2\,(x - 1)^2 + 7$

c) $f_3(x) = 3\,(x + 3)^2 + 3$

Exercise 6.3
Find the vertex (x_0, y_0) of the graphs of the second degree polynomials below. Then, rewrite the formulas to the form $g(x) = ax^2 + bx + c$.

a) $g_1(x) = 2\,(x - 2)^2 + 4$

b) $g_2(x) = -2\,(x - 2)^2 + 4$

c) $g_3(x) = 9\,(x + 1)^2 + 5$

d) $g_4(x) = \frac{1}{3}\,(x + 6)^2 + 12$

Exercise 6.4
Calculate the vertices of each of the second degree polynomials below.

a) $f_1(x) = x^2 - 2x + 2$

b) $f_2(x) = -x^2 + 5$

c) $f_3(x) = -3x^2 + 12x - 13$

d) $f_4(x) = \frac{1}{2}x^2 + 2x + 4$

e) $f_5(x) = -4x^2 + 24x + 35$

f) $f_6(x) = \frac{3}{2}x^2 - 6x + 6$

Exercise 6.5
Find the coefficients a, b and c for each of the equations. Then, calculate d and solve the equations:

a) $2x^2 + 4x - 16 = 0$
b) $-x^2 - 2x + 3 = 0$
c) $2x^2 - 4x + 6 = 0$
d) $4x^2 - 6x - 4 = 0$
e) $2x^2 + 6 = 8x$
f) $2x + 15 = x^2$

Exercise 6.6
Solve the following equations without any tools:

a) $-x^2 + x - 1 = 0$
b) $4x^2 - 2x - 12 = 0$
c) $3x^2 - 6x + 3 = 0$
d) $x^2 - x - 6 = 0$

Exercise 6.7
Solve the equations below—or explain why they have no solution. The zero product rule is useful for some of the equations.

a) $2x^2 = 0$
b) $(x-1)^2 = 0$
c) $7(x+2)^2 = 0$
d) $2x^2 = 8$
e) $x^2 - 5x = 0$
f) $3x^2 - 27x = 0$
g) $2x^2 + 50x = 0$
h) $x^2 + 6 = 0$

Exercise 6.8
Solve the following equations—or explain why they have no solutions:

a) $-5x^2 = 0$
b) $(2x-4)^2 = 0$
c) $-3(x+3)^2 = 0$
d) $3x^2 = 48$
e) $x^2 + \frac{1}{2}x = 0$
f) $3x^2 - 27 = 0$
g) $7x^2 + 51 = 0$
h) $-x^2 + 36 = 0$

Exercise 6.9
Below, you see formulas for a series of second degree polynomials:

a) $f_1(x) = 2x^2 + 3x$
b) $f_2(x) = -3x^2 + x + 1$
c) $f_3(x) = -\frac{3}{4}x^2 + 2x + 4$
d) $f_4(x) = -x^2 + 4$
e) $f_5(x) = 4x^2 + 2x + 3$
f) $f_6(x) = \frac{2}{5}x^2 + x + 1$

The graphs of these functions are parabolas. Order them, so that the steepest parabola opening upwards is first, and the steepest opening downwards is last.

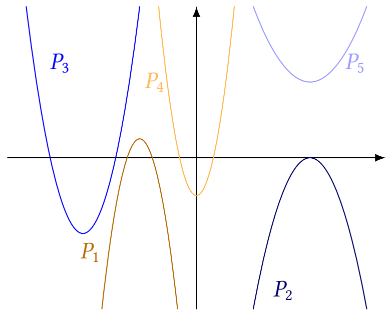
Exercise 6.10
Find the x-axis intercepts and the vertices of the following second degree polynomials:

a) $f_1(x) = 3x^2 - 3$
b) $f_2(x) = 2x^2 + 7x + 2$
c) $f_3(x) = -\frac{1}{4}x^2 - 6x + 5$
d) $f_4(x) = -\frac{1}{3}x^2 - x$
e) $f_5(x) = -3x^2 + 6x - 7$
f) $f_6(x) = \frac{1}{2}x^2 - \frac{1}{2}x - 1$

Exercise 6.11
P_1, P_2, P_3, P_4 and P_5 are graphs of different second degree polynomials, which may be written in the form

$$f(x) = ax^2 + bx + c$$

The discriminant is denoted by d.



In each of the cases, determine the signs of a, b, c and d from the graphs in the figure.

Let a_1 denote the coefficient of the second degree term of P_1, etc. List a_1, a_2, a_3, a_4, a_5 and a_7 in descending order.

Exercise 6.12
Factor the following second degree polynomials:

a) $f(x) = x^2 - x - 30$
b) $g(x) = \frac{1}{2}x^2 - x - 4$
c) $h(x) = 3x^2 + 6x - 429$

Exercise 6.13
Write a formula for the second degree polynomial whose graph passes through the given points:

a) $(1,0), (5,0)$ and $(0,3)$
b) $(-1,0), (3,0)$ and $(1,2)$

Exercise 6.14
Use a CAS to factor the following polynomials, and determine their roots:

a) $x^4 - 3x^3 + 5x^2 - 9x + 6$
b) $x^6 - x^4 - 16x^2 + 16$
c) $2x^5 - 2x^4 - 94x^3 + 202x^2 + 452x - 560$
d) $\frac{1}{2}x^3 - 5x^2 - 16x + 48$

Exercise 6.15
Write a formula for a third degree polynomial with the roots −2, 3 and 7.

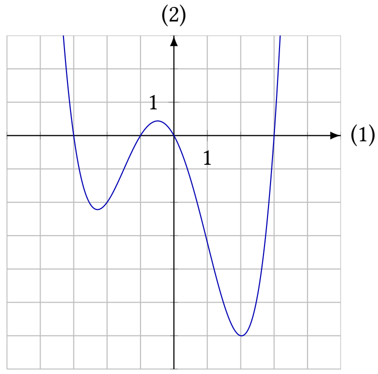
Exercise 6.16
The fourth degree polynomial f has the roots −3, 0, 1 and 5, and the graph of the polynomial passes through $(-1, 8)$.

Determine a formula for this polynomial.

Exercise 6.17
The graph of a third degree polynomial passes through the points $(-1, 0)$, $(0, 3)$, $(6, 0)$ and $(7, 4)$.

Determine a formula for this polynomial.

Exercise 6.18
The figure below shows the graph of the polynomial f.



a) What is the least possible value of the degree of f?

It is given that the degree of f is 4, and the coefficient of x^4 is $\frac{1}{5}$.

b) Determine a formula for f.

7 Trigonometric functions

The *trigonometric* functions are a group of functions which can be used to describe oscillations, but are also used in geometry. In this chapter, we describe the three trigonometric functions *sine* (sin), *cosine* (cos) and *tangent* (tan), and their inverse functions.

The definition of these three functions is based on the so-called *unit circle*. This is a circle with radius 1 and its centre at $(0, 0)$ in a coordinate system, see figure 7.1.

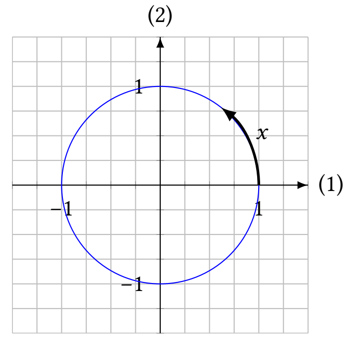


Figure 7.1: The unit circle and the arc length x.

x is in this case not the usual x-coordinate, but rather the length of the arc counter-clockwise from the point $(1, 0)$, see figure 7.1 (if we move clockwise, x is negative).

If we move x along the unit circle, we arrive at the point P. The cosine and the sine are defined to be the first and second coordinates of this point, see figure 7.2. The tangent is the ratio between the sine and the cosine.

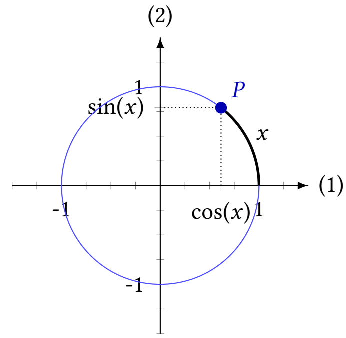


Figure 7.2: The definition of $\cos(x)$ and $\sin(x)$.

So, we define the three functions in this way:

Definition 7.1

Let P be the end point of the arc of length x. Then

1. $\cos(x)$ equals the first coordinate of P.
2. $\sin(x)$ equals the second coordinate of P.
3. $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

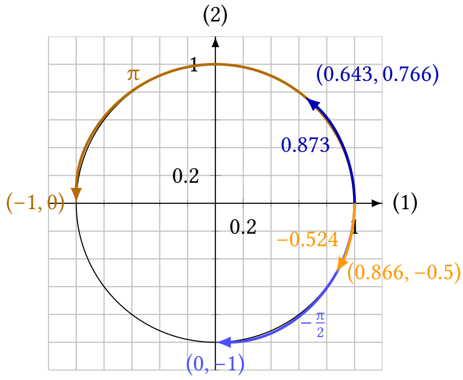
Note that $\tan(x)$ is only defined when $\cos(x) \neq 0$.

The unit circle is a circle with radius 1. We can use the radius to calculate its circumference, which is $2\pi r = 2\pi \cdot 1 = 2\pi$. I.e. if the arc length is π, the arc corresponds to half a circle, and the coordinates of P are $(-1, 0)$. If the arc length is $-\frac{\pi}{2}$, then we have moved a quarter of a circle *clockwise*, and the coordinates of P are $(0, -1)$. Figure 7.3 shows some corresponding arc lengths and coordinates.

Because $\sin(x)$ and $\cos(x)$ are the coordinates of a point on the unit circle, which has radius 1, it follows that both $\sin(x)$ and $\cos(x)$ must have values between -1 and 1, i.e. that

$$-1 \leq \cos(x) \leq 1 \qquad \text{and} \qquad -1 \leq \sin(x) \leq 1\ .$$

Figure 7.3: The figure on the left shows a series of coordinates of end points of a series of different arcs. The table on the right lists the same information, but here the coordinates are shown to be the values of the cosine and the sine of the different values of x.



(a) End points of different arcs.

x	$\cos(x)$	$\sin(x)$
$-\frac{\pi}{2}$	0	−1
−0.524	0.866	−0.5
0.873	0.643	0.766
π	−1	0

(b) Table of corresponding values.

Investigating the symmetry of the unit circle, we may arrive at the following theorem, which we will not prove:

Theorem 7.2

The following identities hold:

1. $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$
2. $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$
3. $\cos(-x) = \cos(x)$
4. $\sin(-x) = -\sin(x)$
5. $\cos(\pi - x) = -\cos(x)$
6. $\sin(\pi - x) = \sin(x)$

Because the radius of the unit circle is 1, we have the following identity between the cosine and the sine, which can be derived from the Pythagorean theorem:

Theorem 7.3: Pythagorean identity

We have

$$\cos(x)^2 + \sin(x)^2 = 1\ .$$

7.1 Graphs of the trigonometric functions

We may treat the cosine and sine as any other mathematical function, e.g. we may draw their graphs. The graphs of these two functions are shown in figure 7.4.

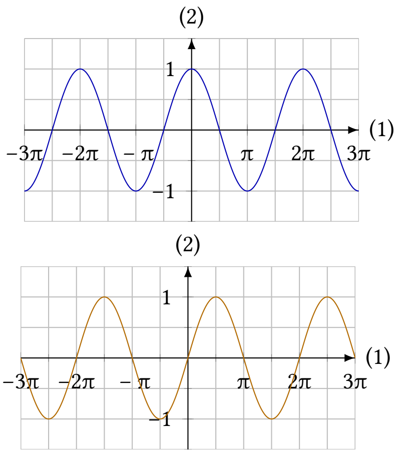


Figure 7.4: The graphs of $\cos(x)$ (above) and $\sin(x)$ (below).

Because the circumference of the unit circle is 2π, we might be tempted to assume that x can only have a value of 2π or less; but this is not true. Values of x larger than 2π simply correspond to more than a whole turn in the circle; while the negative values correspond to clockwise movement. This means that the function values will repeat themselves for each whole turn in the circle—we describe this in more detail in the next section.

On the graphs the x-axis is labeled in units of π. We do this because these are the values of x where the graphs intercept the x-axis or have maxima and minima. If the x-value is a fraction of π, we may also in many cases calculate the exact values of $\cos(x)$ and $\sin(x)$. Some of the exact function values of $\cos(x)$ and $\sin(x)$ are listed in table 7.5.

Table 7.5: Function values of cos and sin.

x	$\cos(x)$	$\sin(x)$
$-\pi$	-1	0
$-\frac{\pi}{2}$	0	-1
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

Periodicity

Looking at the graphs of the two functions, we see that they are *periodic.*

A function is called periodic if the function "repeats itself". The graphs of $\cos(x)$ and $\sin(x)$ show that every time x increases or decreases by 2π, we find the same function values.

The reason for this behaviour is that the cosine and the sine are based on the unit circle, and 2π corresponds to exactly one turn of the circle. Therefore, $\cos(x)$ and $\sin(x)$ will have the same value whenever x increases or decreases by some integer times 2π, i.e.

$$\begin{aligned} \cos(x + k \cdot 2\pi) &= \cos(x) \\ \sin(x + k \cdot 2\pi) &= \sin(x) \qquad \text{where } k \text{ is integer .} \end{aligned}$$

We say that these functions are periodic with *period* $T = 2\pi$. We can also find the period as the distance between two crests of the wave on the graph.

Because these two functions are periodic, they can be used to describe a range of different natural phenomena, which display a repeating pattern, e.g. waves or oscillations.

Tangent

We have yet to mention the tangent. The graph of $\tan(x)$ is shown in figure 7.6. As the figure shows, the function values approach ∞ resp. $-\infty$ whenever x approaches $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$ etc.

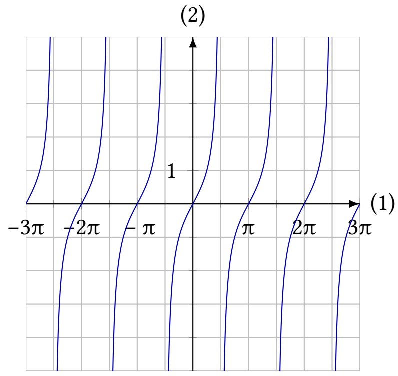


Figure 7.6: The graph of $\tan(x)$.

The reason is that $\tan(x)$ is defined to be

$$\tan(x) = \frac{\sin(x)}{\cos(x)} ,$$

and at these values of x, $\cos(x) = 0$.

As the figure shows, the function $\tan(x)$ is periodic with period π.

7.2 Oscillations

As previously mentioned, the trigonometric functions can be used to describe oscillations. Many oscillations can be described by "wave-shaped" graphs, which look like the graphs of the cosine and the sine (see figure 7.4). Functions that have these types of graph, are of the form $f(x) = a \cdot \sin(bx + c)$. We therefore define a *sine wave* in the following way:

Definition 7.4

A *sine wave* is the graph of a function of the form

$$f(x) = a \cdot \sin(bx + c) ,$$

where $a > 0$, $b > 0$ and c are arbitrary numbers.

We can now investigate how the constants a, b and c affect the appearance of the graph.

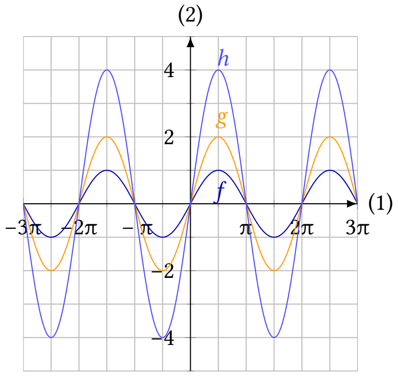


Figure 7.7: The graphs of $f(x) = \sin(x)$, $g(x) = 2\sin(x)$ and $h(x) = 4\sin(x)$.

Figure 7.7 shows the graphs of the three functions

$$f(x) = \sin(x)\ , \quad g(x) = 2\sin(x) \quad \text{and} \quad h(x) = 4\sin(x)\ .$$

The only difference between these functions is the value of the number a. As we can see, the graphs all have the same shape, but not the same height. This means that the number a determines the height of the waves—i.e. the distance from the x-axis to the wave crests—in the graph.

Next, we investigate 3 functions with different values of b. We look at

$$f(x) = 3\sin(x)\ , \quad g(x) = 3\sin\left(\tfrac{1}{2}x\right) \quad \text{and} \quad h(x) = 3\sin(2x)\ .$$

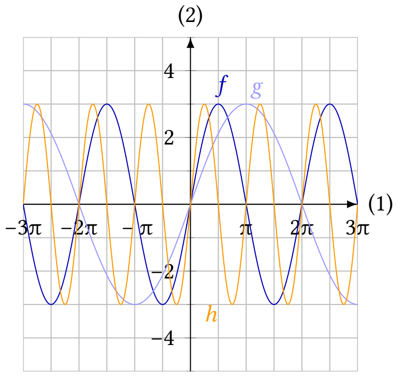


Figure 7.8: The graphs of $f(x) = 3\sin(x)$, $g(x) = 3\sin\left(\frac{1}{2}x\right)$ and $h(x) = 3\sin(2x)$.

As the figure shows, the three graphs have the same wave height—because they all have $a = 3$—but they do not oscillate equally fast, i.e. their *periods* are different.

The function $\sin(x)$ has a period of 2π. I.e. when x increases from 0 to 2π, the graph completes exactly one oscillation. But what about the function $f(x) = a \cdot \sin(bx)$? This function must complete one oscillation, when bx increases from 0 to 2π.

Therefore, we solve the two equations $bx = 0$ and $bx = 2\pi$, and get

$$\begin{aligned} bx &= 0 &\iff&& x &= 0 \\ bx &= 2\pi &\iff&& x &= \frac{2\pi}{b}\ . \end{aligned}$$

So, the period of the function corresponds to an increase in x from 0 to $\frac{2\pi}{b}$. A larger value of b therefore leads to a smaller period, which is also evident in figure 7.8. We now know that the period T is $T = \frac{2\pi}{b}$.

We can also investigate the meaning of the last number, c, by drawing graphs of functions with different values of c, see figure 7.9. The three functions are given by

$$f(x) = 3\sin\left(\tfrac{1}{2}x\right)\ , \quad g(x) = 3\sin\left(\tfrac{1}{2}x + 2\right) \quad \text{and} \quad h(x) = 3\left(\tfrac{1}{2}x - 1\right)\ .$$

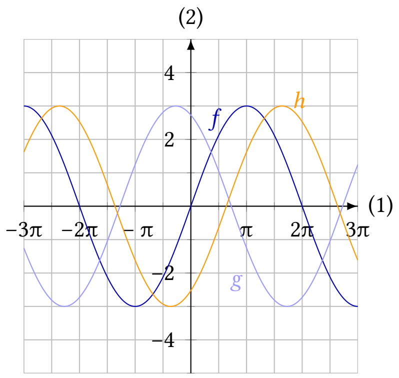


Figure 7.9: Graphs of sine waves with different values of c.

As we can see, the three graphs are horizontal shifts of each other. Because $\sin(x)$ intercepts the x-axis when $x = 0$, the graph of $a \cdot \sin(bx + c)$ intercepts the x-axis when $bx + c = 0$, i.e. when $x = -\frac{c}{b}$. We call this number the *phase*, and it shows where the graph intercepts the x-axis the "first time". Because the graph intercepts the x-axis for each half period, we can find the other intercepts by adding half a period a number of times (or subtract it a number of times).

The period is $T = \frac{2\pi}{b}$, so half a period is $\frac{\pi}{b}$, and the graph therefore intercepts the x-axis at

$$\ldots, \frac{-c - 2\pi}{b}, \frac{-c - \pi}{b}, -\frac{c}{b}, \frac{-c + \pi}{b}, \frac{-c + 2\pi}{b}, \frac{-c + 3\pi}{b}, \ldots\ .$$

We sum up our results in this theorem:

Theorem 7.5

For the function $f(x) = a \cdot \sin(bx + c)$,

1. a is equal to the amplitude, i.e. the distance from the x-axis to the wave crests,
2. the period (i.e. the distance from one wave crest to the next) equals $\frac{2\pi}{b}$, and
3. the phase equals $-\frac{c}{b}$.

Example 7.6 The function f is given by

$$f(x) = 4.5 \cdot \sin(0.43x + 1.2)\ .$$

The graph of the function is shown in figure 7.10.

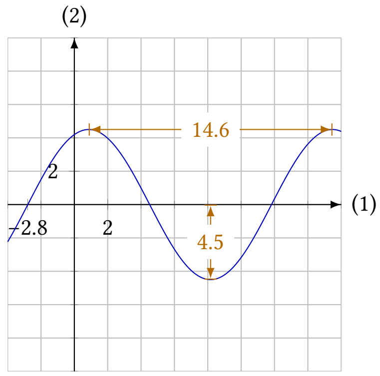


Figure 7.10: The graph of the sine wave $f(x) = 4.5 \cdot \sin(0.43x + 1.2)$.

f has an amplitude of $a = 4.5$, i.e. the height of the waves is 4.5 above the x-axis.

The constant $b = 0.43$, so the period is

$$T = \frac{2\pi}{0.43} = 14.6\ .$$

The phase can be calculated from $c = 1.2$, and we get

$$-\frac{c}{b} = -\frac{1.2}{0.43} = -2.8\ .$$

Therefore, the graph intercepts the x-axis at -2.8. If we want to find the other intercepts, we need to add half a period (i.e. $\frac{1}{2} \cdot 14.6 = 7.3$) or subtract it a number of times.

The cosine

The graph of the cosine is also a wave. But according to theorem 7.2,

$$\cos(x) = \cos(-x) = \cos\left(\frac{\pi}{2} - x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \left(x + \frac{\pi}{2}\right)\right) = \sin\left(x + \frac{\pi}{2}\right)\ .$$

So, the cosine is actually a sine with a phase of $-\frac{\pi}{2}$. Therefore, any oscillation which is described by a cosine, might just as well be decribed by a sine.

7.3 Degrees and radians

The sine and the cosine was defined above using arc lengths in the unit circle. But we could also have defined them using the angles spanned by the arcs. Actually, we could measure the size of an angle by the length of the corresponding arc in the unit circle. When we do this, we say that we measure the angle in *radians*. If we would rather measure the angle in degrees, it is relatively simple to convert one measurement to the other.

A complete circle corresponds, as we know, to an angle of 360°.[1]

Because the circumference of the unit circle is 2π, 2π radians must correspond to 360°. And a right angle (90°) must correspond to $\frac{\pi}{2}$ radians.

[1] There is no mathematical reason for the number 360. It is actually a relic of the ancient babylonian base-60 number system.[2]

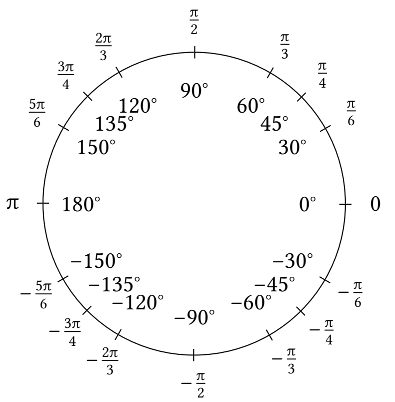


Figure 7.11: The relationship between degrees and radians.

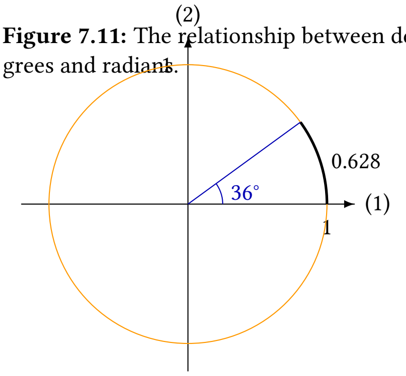


Figure 7.12: 36° corresponds to 0.628 radians.

Figure 7.11 shows corresponding angles measured in degrees and in radians.

Because 2π radians corresponds to a complete circle, and $360°$ is also a complete circle, we get

$$360° = 2\pi \quad \Leftrightarrow \quad 1° = \frac{\pi}{180} \ ,$$

i.e. we can convert degrees to radians by multiplying by $\frac{\pi}{180°}$. And we can convert radians to degrees by multiplying by the inverse fraction $\frac{180°}{\pi}$.

Example 7.7 What is the angle $36°$ in radians?

To answer this we calculate

$$36° \cdot \frac{\pi}{180°} = \frac{36\pi}{180} = \frac{\pi}{5} \approx 0.628 \ .$$

So, $36°$ corresponds to $\frac{\pi}{5}$ radians. I.e. an angle of $36°$ spans an arc of length 0.628 in the unit circle, see figure 7.12.

If we view the trigonometric functions as mathematical functions, we would never use degrees to measure the angles. But trigonometric functions are also used to solve geometric problems, where we use them in formulas involving lengths and angles—and here, it would be natural to measure the angles in degrees, and not in radians.

7.4 Inverse trigonometric functions

In this section, we describe the so-called "inverse trigonometric functions" $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.[2] The three functions are used to solve equations where we know the value of the sine, the cosine or the tangent and the arc length is unknown.

[2]The three functions are also sometimes denoted by arccos, arcsin og arctan, because we use them to find the arc. E.g. arcsin gives the arc corresponding to a certain value of the sine.

In computer programmes or CAS's, the three functions are often denoted by asin, acos and atan.

Example 7.8 To solve the equation $\cos(x) = 0.8$, we use $\cos^{-1}$:

$$\cos(x) = 0.8 \qquad \Leftrightarrow \qquad x = \cos^{-1}(0.8) \ .$$

We use a calculator to find $\cos^{-1}(0.8)$, and get

$$x = \cos^{-1}(0.8) = 0.644 \ .$$

Example 7.9 We solve the equation $\sin(B) = 0.5$ like this:

$$\sin(B) = 0.5 \qquad \Leftrightarrow \qquad B = \sin^{-1}(0.5) = 0.524 \ .$$

As the examples show, we only get one solution. But the cosine and the sine are periodic functions, which means that the equations actually have infinitely many solutions. But of course a calculator can only give us one answer. The question is, which solution do we get?

As it turns out,

1. $\cos^{-1}$ always yields results in the interval from 0 to π.

2. $\sin^{-1}$ always yields results in the intervald from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
3. $\tan^{-1}$ always yields results in the interval from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Therefore, if we want other solutions, we need to think carefully—or solve the equations graphically or using a CAS.

7.5 Equations involving cos and sin

In this section, we show how to find every solution to an equation involving the cosine or the sine. Because these functions are periodic, equations involving them will often have more than one solution—typically infinitely many solutions.

E.g. if we wish to solve the equation $\sin(x) = a$, where a is some number, one of the solutions will be

$$x = \sin^{-1}(a)\ .$$

But—as we mentioned previously—this is only the solution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.[3] Figure 7.13 shows the unit circle, where we see that there is also a solution given by

$$x = \pi - \sin^{-1}(a)\ .$$

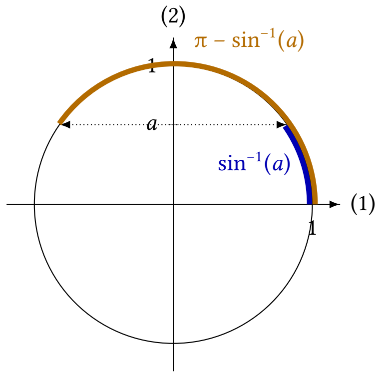


Figure 7.13: The equation $\sin(x) = a$ has two solutions between 0 and 2π.

[3]Corresponding to $-90°$ and $90°$.

Because we can add 2π to x and get the same function values of the sine, the equation $\sin(x) = a$ must have the solutions

$$x = \sin^{-1}(a) + k \cdot 2\pi \quad \vee \quad x = \pi - \sin^{-1}(a) + k \cdot 2\pi\ , \quad k \in \mathbb{Z}\ .$$

Using the same type of argument for the equation $\cos(x)$, we get the following theorem:[4]

[4]Note that we must have $-1 \leq a \leq 1$ because the function values of cos and sin are in this interval.

Theorem 7.10

When $-1 \leq a \leq 1$, we have:

1. The equation $\cos(x) = a$ has the solutions

$$x = \cos^{-1}(a) + k \cdot 2\pi \quad \vee \quad x = -\cos^{-1}(a) + k \cdot 2\pi\ , \quad k \in \mathbb{Z}\ .$$

2. The equation $\sin(x) = a$ has the solutions

$$x = \sin^{-1}(a) + k \cdot 2\pi \quad \vee \quad x = \pi - \sin^{-1}(a) + k \cdot 2\pi\ , \quad k \in \mathbb{Z}\ .$$

Example 7.11 The solutions to the equation

$$\sin(x) = 0.7$$

may be determined graphically by drawing the graph of $\sin(x)$ and the line with equation $y = 0.7$, and then finding the x-coordinates of the intersection points (see figure 7.14).

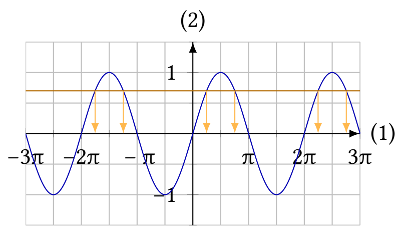


Figure 7.14: The solutions to $\sin(x) = 0.7$ are found by drawing the graph of $\sin(x)$ and the line with equation $y = 0.7$, and finding the x-coordinates of the intersection points.

According to theorem 7.10, this equation has the solutions

$$x = \sin^{-1}(0.7) + k \cdot 2\pi \quad \vee \quad x = \pi - \sin^{-1}(0.7) + k \cdot 2\pi\ , \quad k \in \mathbb{Z}\ ,$$

i.e.[5]

$$x = 0.7754 + k \cdot 2\pi \quad \vee \quad x = 2.3662 + k \cdot 2\pi\,, \quad k \in \mathbb{Z}\,.$$

[5]Here, we calculate the actual values of $\sin^{-1}(0.7)$ and $\pi - \sin^{-1}(0.7)$.

Example 7.12 The equation

$$3\cos(x) - 1 = 0.8$$

can be solved by first isolating $\cos(x)$:

$$3\cos(x) - 1 = 0.8 \quad \Longleftrightarrow \quad 3\cos(x) = 1.8 \quad \Longleftrightarrow \quad \cos(x) = 0.6\,.$$

Next, we use theorem 7.10, and we get

$$x = \cos^{-1}(0.6) + k \cdot 2\pi \quad \vee \quad x = -\cos^{-1}(0.6) + k \cdot 2\pi\,, \quad k \in \mathbb{Z}\,,$$

which reduces to

$$x = 0.9273 + k \cdot 2\pi \quad \vee \quad x = -0.9273 + k \cdot 2\pi\,, \quad k \in \mathbb{Z}\,.$$

Example 7.13 The solutions to the equation

$$\sin(2x - 1) = 0.3$$

may be found by using theorem 7.10. We have $2x - 1$ inside the parenthesis, which means

$$2x - 1 = \sin^{-1}(0.3) + k \cdot 2\pi \quad \vee \quad 2x - 1 = \pi - \sin^{-1}(0.3) + k \cdot 2\pi\,, \quad k \in \mathbb{Z}\,.$$

In both of these equations, we isolate x:

$$x = \frac{\sin^{-1}(0.3) + k \cdot 2\pi + 1}{2} \quad \vee \quad x = \frac{\pi - \sin^{-1}(0.3) + k \cdot 2\pi + 1}{2}\,.$$

We reduce and get

$$x = \frac{1.3047 + k \cdot 2\pi}{2} \quad \vee \quad x = \frac{3.8369 + k \cdot 2\pi}{2}\,,$$

i.e.

$$x = 0.6524 + k \cdot \pi \quad \vee \quad x = 1.9185 + k \cdot \pi\,, \quad k \in \mathbb{Z}\,.$$

Looking at the unit circle and using a similar argument to the one above, we arrive at the following theorem, which we will not prove here:

Theorem 7.14

The equation $\tan(x) = a$ has the solutions

$$x = \tan^{-1}(a) + k \cdot \pi\,, \quad k \in \mathbb{Z}\,.$$

Example 7.15 We solve the equation

$$\tan(x) = 0.5$$

by using theorem 7.14:

$$x = \tan^{-1}(0.5) + k \cdot \pi\,, \quad k \in \mathbb{Z}\,,$$

i.e.

$$x = \frac{\pi}{4} + k \cdot \pi\,, \quad k \in \mathbb{Z}\,.$$

Example 7.16 The equation

$$4\tan(x) + 7 = 10$$

can also be solved using theorem 7.14. We just need to isolate $\tan(x)$ first:

$$4\tan(x) + 7 = 10 \quad\Longleftrightarrow\quad 4\tan(x) = 3 \quad\Longleftrightarrow\quad \tan(x) = \frac{3}{4} \, .$$

We then use the theorem and get

$$x = \tan^{-1}\left(\frac{3}{4}\right) + k \cdot \pi \, , \quad k \in \mathbb{Z} \, ,$$

i.e.

$$x = 0.6435 + k \cdot \pi \, , \quad k \in \mathbb{Z} \, .$$

7.6 Exercises

Exercise 7.1
Calculate the following and illustrate the numbers on the unit circle:

a) $\cos\left(\frac{\pi}{2}\right)$ b) $\sin\left(\frac{\pi}{4}\right)$
c) $\sin(2.7)$ d) $\cos(-5.3)$
e) $\sin(1)$ f) $\cos\left(-\frac{\pi}{3}\right)$

Exercise 7.2
Determine those values of x, where $\cos(x) = \sin(x)$.

Exercise 7.3
Use the Pythagorean identity to determine $\tan(x)$ when

a) $\sin(x) = 0.2$ b) $\cos(x) = 0.6$

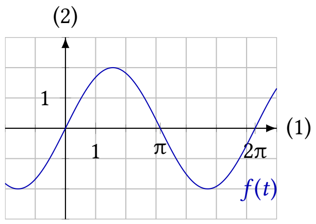
Exercise 7.4
Use the Pythagorean identity to determine $\sin(x)$ when

a) $\cos(x) = 0.4$ b) $\tan(x) = 2$

Exercise 7.5
The graph of the function f is shown below. The formula for f is of the form

$$f(t) = A\sin(\omega t + \phi)$$

Use the graph to determine the constants A, ω and ϕ.

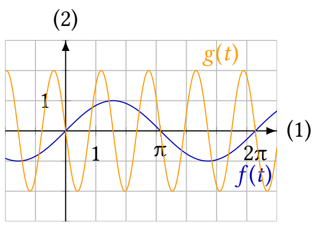


Exercise 7.6
In the following questions, the graph of $f(t) = \sin(t)$ is shown along with the graphs of two other functions g and h.

a) The graph of the function g is shown below. The formula for g has the form

$$g(t) = A\sin(\omega t + \phi)$$

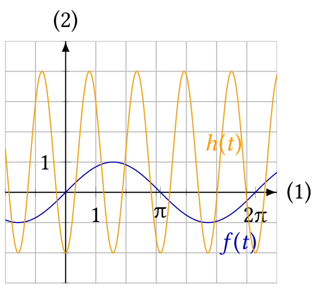
Use the graph to determine the constants A, ω and ϕ.



b) The graph of the function h is shown below. The formula for the function is of the form

$$h(t) = A\sin(\omega t + \phi) + b$$

Use the graph to determine the constants A, ω, ϕ and b
(Note that the period of h is π).



Exercise 7.7
Draw the graph of each function, and determine the amplitude and the period of each:

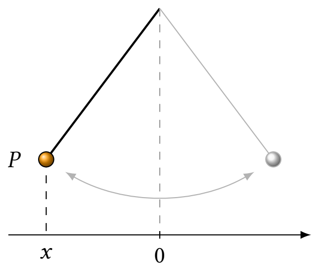
a) $f(x) = 2 \cdot \sin(\frac{1}{3}x + 1)$

b) $g(x) = \frac{1}{4} \cdot \sin(3x - 2)$

c) $h(x) = 3 \cdot \sin(4x + \frac{\pi}{2})$

d) $k(x) = 5 \cdot \sin(\frac{1}{2}x + \pi)$

Exercise 7.8
The figure shows a swinging pendulum.



The horizontal position of the pendulum is given by

$$x = 4 \cdot \sin\left(\tfrac{1}{3}t - 1\right), \quad 0 < t < 10\pi$$

a) Determine the maximum and minimum value of x.

b) Determine the amplitude and the period.

c) Determine for which intervals of time, $x > 3$.

Exercise 7.9
In the interval $[0; 2\pi]$, solve the equation $\sin(x) = 0.731$.

Illustrate the solution in the unit circle.

Exercise 7.10
In the interval $[-\pi; \pi]$, determine the solutions to the equation

$$\cos(2t) = 0.634$$

Exercise 7.11
Find all of the solutions to the equations:

a) $\cos(3x) = 0.7$ b) $3\sin(2x - 1) = 1.5$

Exercise 7.12
Solve each of the equations:

a) $\tan(x) = -1.72$ b) $\frac{1}{\tan(x)} = 0.27$

Set theory A

Set theory is one of the fundamental areas of mathematics which is used in almost every other area (e.g. the concept of a function, or probability theory). In this appendix, we will describe briefly some of the central concepts of set theory.

A.1 Sets

A mathematical set can be defined as a collection of objects. In principle, an "object" may be anything, but here we will restrict ourselves to sets of numbers.

A *set* can be defined this way:

Definition A.1

A *set* is a well-defined collection of mutually different objects.

Note that the objects in the set have to be different. Thus, we cannot create a set of four 2's; there can be only one.

Sets are usually denoted by uppercase letters, e.g. the set S. If we want to show that this set consists of the numbers 1, 2, 3, and 10, we write

$$S = \{1, 2, 3, 10\} \ . \tag{A.1}$$

The curly brackets { } are also called "set brackets" or "braces".[1]

[1]When we write a set like this, we do not need to list the numbers in any particular order, i.e. $\{1, 2, 3\}$ and $\{3, 1, 2\}$ is the same set.

We see that the set includes the number 3. If we want to state this fact, we write

$$3 \in S \, ,$$

which is read "3 is in S" or "3 is an element of S".

If we want to say the exact opposite of this, we can negate the symbol by crossing it out. Therefore, if we want to state that 7 is not an element of S, we write

$$7 \notin S \, .$$

Large sets

Sometimes sets contain many numbers. If this is the case, it might be overwhelming to list all the numbers like we did in (A.1), and it might also be quite impossible to get a clear idea of the set. E.g. if we want to write the set H of every positive integer between 0 and 100, we may instead write

$$H = \{1, 2, 3, \dots, 100\} \ .$$

We write just enough numbers for the reader to discern the pattern, and then we write the last number. The dots … are then written instead of all the numbers in between.

We may even have sets that are "limitless". The set of every positive, odd number contains infinitely many elements. We may write this set as

$$O = \{1, 3, 5, 7, \dots\} \ .$$

Special sets

Some sets are used so often in set theory that we define special symbols for them. These sets are

$\emptyset$ *the empty set*, which is a set containing no elements,

$\mathbb{N}$ *the natural numbers*, the set of all positive integers, $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ (note that 0 is not an element of this set),

$\mathbb{Z}$ *the integers*, the set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$,

$\mathbb{Q}$ *the rational numbers*, the set of all numbers which may be expressed as fractions, e.g. $\frac{1}{7}$ and $-\frac{5}{2}$, and

$\mathbb{R}$ *the real numbers*, the set containing every number on the number line. Some of the numbers in $\mathbb{R}$ are -1, $\frac{1}{6}$, π, e and $\sqrt{2}$.

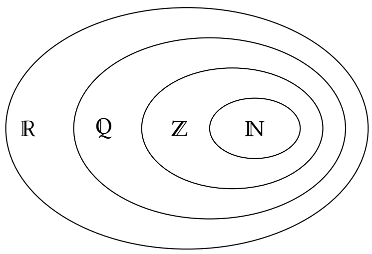


Figure A.1: Every natural number is also an integer, and every integer is a rational number, etc.

It is noteworthy that every natural number is also an integer, i.e. the set $\mathbb{N}$ is a part of the set $\mathbb{Z}$. In the same way, the set of integers $\mathbb{Z}$ is a part of the set of rational numbers $\mathbb{Q}$,[2] and the set of rational numbers is a part of the set of real numbers (see figure A.1).

[2] The reason is that every integer may be written as a fraction, e.g. $2 = \frac{6}{3}$ and $-4 = -\frac{8}{2}$.

A.2 Set-builder notation

It is sometimes useful to describe a set by giving a condition (or more) which the elements of the set satisfy. An example of this might be the set

$$A: \ \text{every number between 1 and 6} \ .$$

We can write this using the so-called set-builder notation

$$A = \{x \in \mathbb{R} \mid 1 < x < 6\} \ .$$

This expression is comprised of two parts. The part before the vertical line ($x \in \mathbb{R}$) tells us from which larger set, the numbers in our set are taken. Here, $x \in \mathbb{R}$ means that the numbers come from the set of real numbers, i.e. we are allowed to use any number imaginable. The part after the line is the condition these numbers have to satisfy; in this case the condition is that the numbers must be between 1 and 6.[3]

[3] $1 < x < 6$ is a contraction of $1 < x$ and $x < 6$, i.e. we are looking for numbers which are simultaneously greater than 1 and less than 6.

Other examples of sets written using this notation are

$$B = \{x \in \mathbb{Z} \mid 1 < x < 6\} \ , \qquad C = \left\{x \in \mathbb{Q} \,\middle|\, x^2 < 9\right\} \ .$$

Here, B is the set of integers between 1 and 6. This set may be written as a list:

$$B = \{2, 3, 4, 5\} \ .$$

C is the set of fractions whose squares are less than 9. We *cannot* list this set, because infinitely many numbers satisfy this condition.

A.3 Intervals

The set A in the previous section is an example of what we call an *interval*. An interval is a set which contains every real number between two given values, e.g. "the set of every real number between 1 and 6" or "every real number from −5 up to and including 80". Sets containing every number greater than or less than a given value are also called intervals.

We often need to talk about intervals, and therefore a notation has been invented which makes it easier to write intervals. E.g. the set A above may be written as

$$A = \ ]1; 6[\ .$$

This notation means that the set A consists of every number between 1 and 6. The numbers 1 and 6 are called the *left* and *right end values*. Because the brackets face away from the numbers, neither 1 nor 6 is included in the interval (see figure A.2).

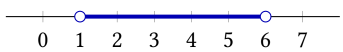


Figure A.2: The interval $]1; 6[$. The empty circles at 1 and 6 show that these numbers are not included in the interval.

If we want to write the interval from 1 to 6 including 1 and 6, we write instead

$$D = [1; 6] \ .$$

We can also write D in set-builder notation as $D = \{x \in \mathbb{R} \mid 1 \leq x \leq 6\}$.

A few more examples are (see figure A.3).

$$]-3; 2] = \{x \in \mathbb{R} \mid -3 < x \leq 2\}$$
$$\left[-4; \tfrac{1}{2}\right[\ = \left\{x \in \mathbb{R} \,\middle|\, -4 \leq x < \tfrac{1}{2}\right\} \ .$$

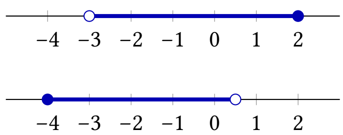


Figure A.3: The intervals $]-3; 2]$ and $\left[-4; \frac{1}{2}\right[$.

If we want to write the interval containing every number larger than 3, we use the symbol ∞ (infinity) as the right end value:

$$E = \ ]3; \infty[\ .$$

Thus, the set E contains every number larger than 3. If we instead want to talk about, e.g., every number less than or equal to 5, we write

$$F = \ ]-\infty; 5] \ .$$

Note that when we use the symbol ∞, the bracket has to face away from the symbol (this is because ∞ is not a number in itself, it merely shows that the interval does not end in this direction).

If we let the interval be unlimited in both directions, we get the interval consisting of every number, i.e.

$$\mathbb{R} = \ ]-\infty; \infty[\ .$$

A.4 Set operations

We may combine two given sets A and B in different ways to form new sets. E.g. we may look at the numbers which are an element of both A and B, or the numbers which are elements of A, but not B.

The following definitions demonstrate some of the ways in which we may form new sets (so-called set operations).

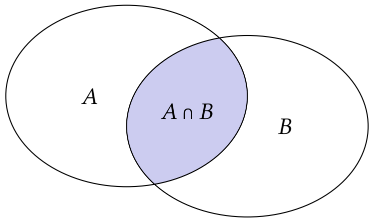


Figure A.4: The intersection $A \cap B$.

Definition A.2

The *intersection* of two sets A and B contains every number, which is an element of both A and B. We denote the intersection of A and B by $A \cap B$.

Example A.3 If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$, then

$$A \cap B = \{2, 4\} \ .$$

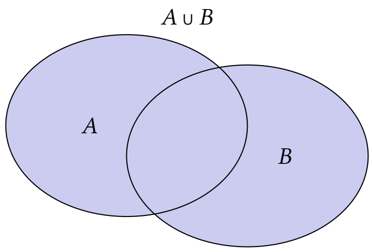


Figure A.5: The union $A \cup B$.

Definition A.4

The *union* of two sets A and B contains every number which is an element of either A or B (or both). The union of A and B is denoted by $A \cup B$.

Example A.5 If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8\} \ .$$

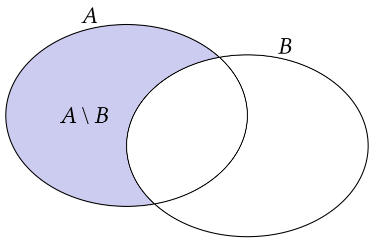


Figure A.6: The difference $A \setminus B$.

Definition A.6

The *difference* between two sets A and B contains every number which is an element of A, but not of B. We denote the difference between A and B by $A \setminus B$.

Example A.7 If $A = \{1, 2, 3, 4, 5\}$ og $B = \{2, 4, 6, 8\}$, then

$$A \setminus B = \{1, 3, 5\} \ .$$

and

$$B \setminus A = \{6, 8\} \, .$$

Thus, when we look at differences between sets, the order matters.

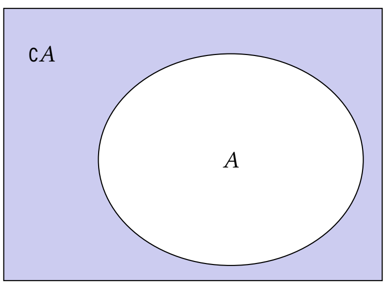


Figure A.7: The complement $\complement A$.

Lastly, we define the *complement*, which contains all of the numbers which are not elements of a given set. If this concept is to make sense, we first need to define a universal set, which contains all of the numbers we will allow any given set to contain.[4]

[4]The universal set might be the set of all numbers, i.e. $\mathbb{R}$, but it could also be the natural numbers $\mathbb{N}$, or some other given set such as, e.g., $\left\{-2, 0, \frac{1}{3}, 7\right\}$.

Definition A.8

Let U be the *universal set*. The *complement* $\complement A$ contains every element of U which is not an element of A, i.e. $\complement A = G \setminus A$.

A.5 Relations between sets

It is sometimes important to be able to compare different sets. E.g. we would like to know when two sets are equal.

Definition A.9

Two sets A and B are said to be equal when they contain the exact same elements. In this case, we write $A = B$.

If all of the elements of A are elements of B, but all of the elements of B are not necessarily elements of A, we call A a subset of B.

Definition A.10

The set A is said to be a *subset* of the set B, if every element of A is also an element of B. We write $A \subseteq B$.

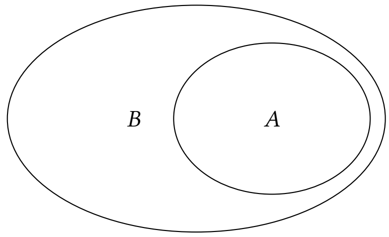


Figure A.8: A is a subset of B, $A \subseteq B$.

Example A.11 If two sets are given by

$$A = \{-1, 1\} \quad \text{and} \quad B = \{-2, -1, 0, 1, 2, 3, 4\} \ ,$$

then A is a subset of B, i.e. $A \subseteq B$.

Previously, we have demonstrated that every natural number is an integer, and that every integer is a rational number, etc. We may express this using the concept of subsets, and write

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \ .$$

Figure A.1 illustrates this.

If two sets have no common element, we call them *disjoint*.

Definition A.12

Two sets A and B are called *disjoint* when no element of A is an element of B (and no element of B is an element of A), i.e. when $A \cap B = \emptyset$.

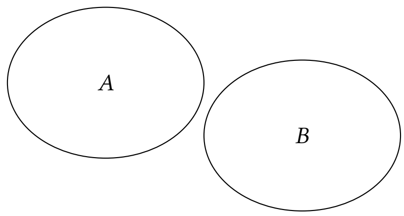


Figure A.9: A and B are disjoint sets.

Example A.13 The sets $A = \{1, 2, 3\}$ and $B = \{-1, 0, 7\}$ are disjoint sets.

A.6 Exercises

Exercise A.1
List the following sets (in braces):

a) The set of every integer from and including 5 up to and including 10.
b) The sets of every square between 1 and 100.
c) The set of every integer which satisfies $x^2 \leq 9$.
d) The set of every prime from and including 17 up to and not including 43.

Exercise A.2
List the following sets (in braces):

a) $A = \{x \in \mathbb{N} \mid x \text{ divides } 12\}$
b) $B = \{x \in \mathbb{Z} \mid x \text{ divides } 12\}$
c) $C = \{x \in \mathbb{N} \mid 4 \text{ divides } x\}$
d) $D = \{y \mid y \text{ is a prime less than } 17\}$
e) $E = \{z \mid z \text{ is a two-digit square}\}$
f) $F = \left\{x \in \mathbb{Z} \,\middle|\, x^2 \text{ is even}\right\}$

Exercise A.3
Determine which of the following statements are true:

a) $3 \in \mathbb{Z}$
b) $3 \notin \mathbb{Z}$
c) $-\frac{1}{2} \in \mathbb{Z}$
d) $10.3 \in \mathbb{N}$
e) $\sqrt{3} \in \mathbb{R}$
f) $-\sqrt{4} \in \mathbb{Z}$
g) $\sqrt{\frac{9}{16}} \notin \mathbb{Q}$
h) $\frac{121}{11} \in \mathbb{N}$
i) $0 \in \mathbb{N}$
j) $\sqrt{3^2 + 4^2} \in \mathbb{Q}$

Exercise A.4
Determine the elements of the following sets:

a) $A = \{x \mid 4x = 1\}$
b) $B = \left\{x \in \mathbb{N} \,\middle|\, x(x-1) + 2x = x^2 + 7\right\}$
c) $C = \left\{x \in \mathbb{Z} \,\middle|\, (x+2)(x-2) = 3x^2 - 22\right\}$
d) $D = \left\{x \,\middle|\, 2x - 8 + \frac{1}{2}(x-3) = -x\right\}$

Exercise A.5
Write the following sets as intervals:

a) $]23; 44] \setminus [31; 56]$
b) $[32; 89[\;\setminus\;]76; 97]$
c) $]-4; 7] \cup [3; 12]$
d) $[-8; -5] \cap\;]-6; 10[$
e) $[-1; 2] \setminus\;]0; \infty[$
f) $]3; 7[\;\cap [4; 13[$

Exercise A.6
Let

$$A = \{1, 2, 3, 4\}$$
$$B = \{-2, -1, 0, 1, 2, 3\}$$
$$C = \{-2, 0, 2, 4, 6, 8\}$$

and determine:

a) $A \cap B$
b) $A \cup B$
c) $A \cap C$
d) $B \cup C$
e) $A \setminus B$
f) $B \setminus A$
g) $(A \cap B) \cup C$
h) $C \setminus A$
i) $(A \cup B) \cap C$
j) $(A \cap B) \cup (A \cap C)$

Bibliography

[1] Ole B. Andersen et al. *Den dynamiske jord. Sumatrajordskælvet flyttede videnskaben*. Danmarks Rumcenter og GEUS. URL: http://www.geus.dk/viden_om/ddj/ddj.pdf.

[2] Petr Beckmann. *A History of Pi*. Barnes & Noble, Inc., 1971.

[3] Victor J. Katz. *A History of Mathematics. An Introduction*. 2nd ed. Addison-Wesley Educational Publishers, Inc., 1998.

[4] United States Census Bureau. *Demographic Overview — Custom Region — Honduras*. URL: http://www.census.gov/population/international/data/idb/region.php?N=%20Results%20&T=13&A=separate&RT=0&Y=2014&R=-1&C=HO (visited on 03/05/2014).