MATHEMATICS B_1

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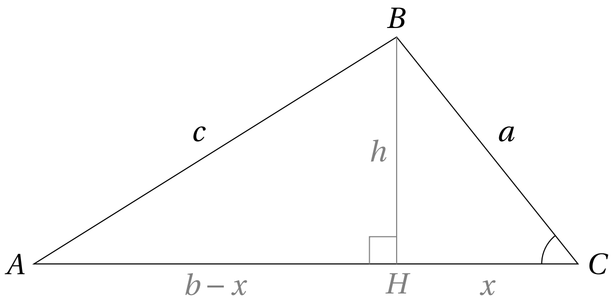

Mathematics B1
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Preface

This document is a translation of the Danish "Matematik B1", which is a textbook on B level mathematics of the Danish stx. Since English is not my first language, I apologise in advance for errors in translation.

The primary aim is to provide a textbook without too much "clutter". Examples are kept to a minimum, and the text mainly covers the basic mathematics. It would therefore be a good idea to supplement the text with examples and other materials that cover specific uses of the mathematical tools.

Mike Auerbach

ORIGINAL PREFACE (IN DANISH)

Disse matematiknoter dækker kernestoffet (og en smule mere) for det første halve år i et studieretningsforløb på B-niveau på stx. Noterne er skrevet med det formål at have en grundbog, som kun indeholder den grundliggende matematiske teori. I forbindelse med samarbejde i studieretningen eller med andre fag er det derfor nødvendigt at supplere med eksempler og andet materiale, der dækker konkrete anvendelser.

Til gengæld dækker noterne den rent matematiske fremstilling af kernestoffet på stx, hvilket ifølge min opfattelse gør dem velegnede til en første behandling af stoffet samt i forbindelse med eksamenslæsningen.

Til slut en stor tak til de mange matematikkolleger, der er kommet med rettelser og gode ændringsforslag. De fejl og mangler, der stadig måtte findes, er naturligvis udelukkende mit ansvar.

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1 Logarithms

1.1 THE COMMON LOGARITHM

The *common logarithm* is a function that is used to solve equations of the form $10^x = c$ where c is a constant. The function log is defined as follows:

Definition 1.1

The number $\log(c)$ is the number that solves the equation $10^x = c$.

From definition 1.1 we can deduce that

$$\begin{aligned} \log(10) &= 1 && \text{since } 10^1 = 10 \\ \log(100) &= 2 && \text{since } 10^2 = 100 \\ \log(1000) &= 3 && \text{since } 10^3 = 1000 \\ &\vdots \\ \log(10^n) &= n\,. \end{aligned}$$

In the same way that $\sqrt{x}$ is the opposite of x^2, $\log(x)$ is the opposite of 10^x. If we solve the equation $x^2 = 49$, we obtain the solution $x = \sqrt{49}$. The value of $\sqrt{49}$ can be determined from the graph of $y = x^2$. We start at 49 on the y-axis and move to the x-axis via the graph. As expected, we find $\sqrt{49} = 7$.

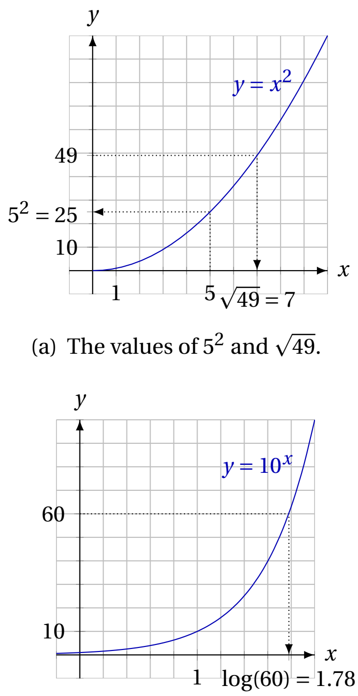
The same method can be used to solve the equation $10^x = 60$. Here, the graph of $y = 10^x$ is used. We find 60 on the y-axis and move to the x-axis via the graph. The value of $\log(60)$ is then found to be $\log(60) = 1.78$ (cf. figure 1.1).

It would be tedious to draw the graph of $y = 10^x$ every time we needed to solve this sort of equation. Luckily, most calculators have a log button.

Example 1.2
To solve the equation $10^x = 6$, we use the following method:

$$10^x = 6 \quad \Leftrightarrow \quad x = \log(6) \quad \Leftrightarrow \quad x = 0.7782\,.$$

The number $\log(6)$ is found using a calculator.



(a) The values of 5^2 and $\sqrt{49}$.

(b) The value of $\log(60)$.

Figure 1.1: In the same way that $\sqrt{x}$ is the opposite of x^2, $\log(x)$ is the opposite of 10^x.

Using the method in the example, we can solve equations involving 10^x—but what about 2^x or 7^x? To solve equations involving these terms, we use the following theorem, the proof of which can be found in section 1.1 below.

Theorem 1.3

The solution to the equation $a^x = c$ is $x = \frac{\log(c)}{\log(a)}$.

Example 1.4
The equation $4^x = 23$ can be solved like this:

$$4^x = 23 \qquad \Leftrightarrow \qquad x = \frac{\log(23)}{\log(4)} \qquad \Leftrightarrow \qquad x = 2.262\,.$$

Logarithm Rules

Since the common logarithm is the opposite of 10^x, we find that[1]

$$\log(10^x) = x \qquad \text{and} \qquad 10^{\log(x)} = x\,.$$

We use this to prove the following:

Theorem 1.5

The following holds for the common logarithm:

1. $\log(a \cdot b) = \log(a) + \log(b)$.
2. $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$.
3. $\log(a^r) = r \cdot \log(a)$.

Proof
To prove these three rules, we use the facts $a = 10^{\log(a)}$ and $b = 10^{\log(b)}$.[2]

1. For $\log(a \cdot b)$ we have

$$\log(a \cdot b) = \log\left(10^{\log(a)} \cdot 10^{\log(b)}\right) = \log\left(10^{\log(a)+\log(b)}\right) = \log(a) + \log(b)\,.$$

2. For $\log\left(\frac{a}{b}\right)$ we have

$$\log\left(\frac{a}{b}\right) = \log\left(\frac{10^{\log(a)}}{10^{\log(b)}}\right) = \log\left(10^{\log(a)-\log(b)}\right) = \log(a) - \log(b)\,.$$

3. For $\log(a^r)$ we have

$$\log(a^r) = \log\left(\left(10^{\log(a)}\right)^r\right) = \log\left(10^{r\cdot\log(a)}\right) = r \cdot \log(a)\,.$$

This proves the theorem. ■

[1] This is because e.g. $\log(10^2)$ is a solution to the equation $10^x = 10^2$, which has the solution $x = 2$. A similar calculation can be made for all positive numbers.

[2] We also use that

1. $10^n \cdot 10^m = 10^{m+n}$,
2. $\frac{10^n}{10^m} = 10^{n-m}$ and
3. $(10^n)^m = 10^{n \cdot m}$.

Example 1.6
Using theorem 1.5, we calculate

$$\log(10x) = \log(10) + \log(x) = 1 + \log(x)\,,$$
$$\log\left(\frac{x}{10}\right) = \log(x) - \log(10) = \log(x) - 1\,,$$
$$\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -1 \cdot \log(x) = -\log(x)\,.$$

Rule 3 in theorem 1.5 has many uses. Here, we use it to prove theorem 1.3:

Proof (of theorem 1.3)
We want to solve $a^x = k$. If we use log on both sides of the equation, we get

$$a^x = k \qquad \Leftrightarrow \qquad \log(a^x) = \log(k)\,.$$

Using rule 3 from theorem 1.5, the equation can be rewritten:

$$x \cdot \log(a) = \log(k) \qquad \Leftrightarrow \qquad x = \frac{\log(k)}{\log(a)}\,.$$

This proves the theorem. ∎

1.2 THE NATURAL LOGARITHM

Euler's Number

The number e—also known as *Euler's number*—plays a role in many areas of mathematics. Like π, it is an irrational number.[3] This means that we cannot write the number e with a finite amount of decimals. The first 24 decimals of e are

$$\mathrm{e} = 2.718\,281\,828\,459\,045\,235\,360\,287\ldots\,.$$

The number e is used to define the natural logarithm.

[3] An irrational number is a number that cannot be written as a fraction. Irrational numbers are characterised by an infinite number of decimals with no recurring pattern.

The Natural Logarithm

Just as the common logarithm (log) is defined in terms of the number 10, the *natural logarithm* is defined in terms of the number e.

Definition 1.7

The number $\ln(c)$ is the number that solves the equation $\mathrm{e}^x = c$.

$\ln(c)$ is called the *natural logarithm* of c.

From this definition, we deduce that

$$\ln(\mathrm{e}^x) = x \qquad \text{and} \qquad \mathrm{e}^{\ln(x)} = x\,.$$

Since the natural logarithm is defined in a similar way as the common logarithm, similar rules apply:

Theorem 1.8

For the natural logarithm the following holds:

1. $\ln(a \cdot b) = \ln(a) + \ln(b)$.
2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.
3. $\ln(a^r) = r \cdot \ln(a)$.

This theorem can be proven in the same manner as theorem 1.5. Therefore the proof is left as an exercise for the reader.

The Relation Between log and ln

It turns out that there is a simple relation between the two logarithmic functions log and ln. Since $x = \mathrm{e}^{\ln(x)}$ we have:[4]

$$\log(x) = \log(\mathrm{e}^{\ln(x)}) = \ln(x) \cdot \log(\mathrm{e}) \ .$$

From this we get

$$\log(x) = \log(\mathrm{e}) \cdot \ln(x) \ .$$

Now we look at the expression $\frac{\log(a)}{\log(b)}$. We get

$$\frac{\log(a)}{\log(b)} = \frac{\log(\mathrm{e}) \cdot \ln(a)}{\log(\mathrm{e}) \cdot \ln(b)} = \frac{\ln(a)}{\ln(b)} \ .$$

Here, we see that when dividing logarithms, it does not matter if we use log or ln, as long as we do not mix them.

[4] The last equality follows from rule 3 in theorem 1.5.

1.3 EXPONENTIAL FUNCTIONS

The exponential function $f(x) = b \cdot a^x$ is often written in another way. Since $a = \mathrm{e}^{\ln(a)}$, the exponential function can be written as

$$f(x) = b \cdot a^x = b \cdot \left(\mathrm{e}^{\ln(a)}\right)^x = b \cdot \mathrm{e}^{\ln(a) \cdot x} \ .$$

Thus we write

$$f(x) = b \cdot \mathrm{e}^{cx} \quad (\text{where } c = \ln(a)) \ .$$

Example 1.9

The exponential function $y = 4.6 \cdot 9.1^x$ can also be written as

$$f(x) = 4.6 \cdot \mathrm{e}^{2.2x}$$

because $\ln(9.1) = 2.2$.

As we already know, the exponential function $f(x) = b \cdot a^x$ is increasing if $a > 1$, and decreasing if $0 < a < 1$. Since $c = \ln(a)$, it follows that the exponential function $f(x) = b \cdot \mathrm{e}^{cx}$ is

1. increasing if $c > 0$, and
2. decreasing if $c < 0$.

Because of this, we sometimes write exponential functions as

1. $f(x) = b \cdot \mathrm{e}^{cx}$ when the function is increasing, and
2. $f(x) = b \cdot \mathrm{e}^{-cx}$ when the function is decreasing.

Doing this ensures that c is always a positive number.

There are several good reasons for writing exponential functions in the way just described. One of the reasons has to do with calculations using units. If the function has the form $f(x) = b \cdot a^x$ it is not possible to assign a meaningful unit to the number a. On the other hand, in $f(x) = b \cdot \mathrm{e}^{cx}$, the unit of c can be chosen in such a way that it exactly cancels out the unit of x.

Doubling Time and Half Life

For an increasing exponential function, the doubling time T_2 is given by the formula

$$T_2 = \frac{\ln(2)}{\ln(a)} .$$

If the increasing exponential function is of the form $f(x) = b \cdot \mathrm{e}^{cx}$, the formula becomes[5]

[5]Here we use that $c = \ln(a)$.

$$T_2 = \frac{\ln(2)}{c} \qquad (\text{when } f(x) = b \cdot \mathrm{e}^{cx}) .$$

If a decreasing exponential function is of the form $y = b \cdot \mathrm{e}^{-cx}$ (notice the sign of c), we have $-c = \ln(a)$. Therefore the half life is[6]

[6]That $\frac{1}{2} = 2^{-1}$ follows from the rule of negative exponents, $a^{-n} = \frac{1}{a^n}$.

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{-c} = \frac{\ln(2^{-1})}{-c} = \frac{-1 \cdot \ln(2)}{-c} = \frac{\ln(2)}{c} .$$

This means the half life can be calculated using the formula

$$T_{\frac{1}{2}} = \frac{\ln(2)}{c} \qquad (\text{when } f(x) = b \cdot \mathrm{e}^{-cx}) .$$

2 Power Functions

Definition 2.1

A power function is a function of the form

$$f(x) = b \cdot x^a, \qquad x > 0\,,$$

where a and b are constants and $b > 0$.

Notice that power functions are only defined for positive x. This is because values of a exist where x^a cannot be defined for all numbers x.[1]

[1] An example is x^{-1}, which is the same as $\frac{1}{x}$. Since we cannot divide by 0, this function is undefined for $x = 0$.

Since we also require $b > 0$, the graph of a power function exists only in the first quadrant, i.e. the x- and y-coordinates are both positive.

Because of this, the graph does not intercept the y-axis, since it is not defined for $x = 0$. But from the definition, we instead find that the graph of a power function will always pass through the point $(1, b)$ since

$$f(1) = b \cdot 1^a = b\,.$$

A few examples of relations involving power functions are

Example 2.2
The area A of a circle with radius r is

$$A = \pi \cdot r^2\,.$$

Here, the radius and the area are connected by a power function. The constants a and b in the definition have the values $a = 2$ and $b = \pi$.

Example 2.3
The speed v (in km/h) of a tsunami wave is a power function of the sea depth d (in metres),[1]

$$v = 11.2 \cdot d^{0.5}\,.$$

Here $a = 0.5$ and $b = 11.2$.

2.1 THE GRAPH OF A POWER FUNCTION

The number a in definition 2.1 is called the *exponent*. This number determines the shape of the graph of the power function. In figure 2.1 we see how the shape of the graph changes for different values of a. We have the following theorem

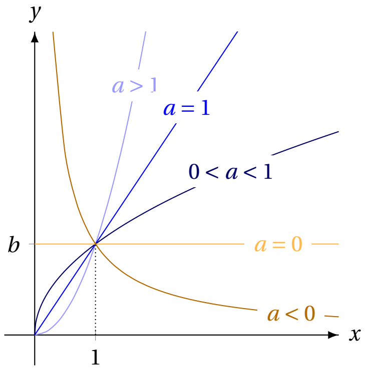


Figure 2.1: The possible shapes of the graph of a power function.

Theorem 2.4

For a power function $f(x) = b \cdot x^a$ we have

1. If $a > 0$ the function is increasing.
2. If $a < 0$ the function is decreasing.

From two points on the graph of a power function, it is possible to determine the numbers a and b (see figure 2.2.)

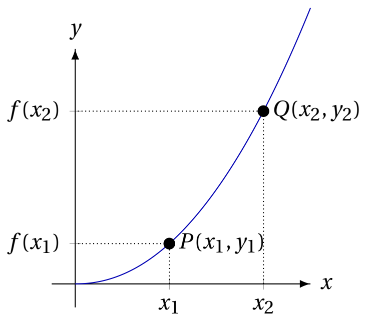


Figure 2.2: Two points on the graph of a power function.

Theorem 2.5

If the graph of a power function $f(x) = b \cdot x^a$ passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ then

$$a = \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{x_2}{x_1}\right)} \qquad \text{and} \qquad b = \frac{y_1}{x_1^a}\,.$$

Proof

If the points $P(x_1; y_1)$ og $Q(x_2; y_2)$ are on the graph of $f(x) = b \cdot x^a$ then

$$\begin{aligned} y_2 &= b \cdot x_2^a \\ y_1 &= b \cdot x_1^a \end{aligned} \tag{2.1}$$

If we divide these two equations, we get

$$\begin{aligned} \frac{y_2}{y_1} &= \frac{b x_2^a}{b x_1^a} && \Leftrightarrow \\ \frac{y_2}{y_1} &= \frac{x_2^a}{x_1^a} && \Leftrightarrow \\ \frac{y_2}{y_1} &= \left(\frac{x_2}{x_1}\right)^a . \end{aligned}$$

This is an exponential equation. In order to solve it, we therefore use the common logarithm on both sides of the equation and get

$$\begin{aligned} \log\left(\frac{y_2}{y_1}\right) &= \log\left(\left(\frac{x_2}{x_1}\right)^a\right) && \Leftrightarrow \\ \log\left(\frac{y_2}{y_1}\right) &= a \cdot \log\left(\frac{x_2}{x_1}\right) && \Leftrightarrow \\ \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{x_2}{x_1}\right)} &= a\,. \end{aligned}$$

This proves the formula for a.

To prove the formula for b, we look once more at equation (2.1):

$$y_1 = b \cdot x_1^a \quad \Leftrightarrow \quad \frac{y_1}{x_1^a} = b\,.$$

We have now also proven the formula for b. ■

2.2 GROWTH OF A POWER FUNCTION

A power function grows in such a way that whenever the independent variable is multiplied by a fixed number, the dependent variable is multiplied by a corresponding fixed number. We have the following theorem:

Theorem 2.6

Let f be the power function $f(x) = b \cdot x^a$. If a value of x is multiplied by the number k, the corresponding value of the function $f(x)$ is multiplied by k^a.

Proof
If x is multiplied by k, the new value of the function will be $f(k \cdot x)$. But

$$f(k \cdot x) = b \cdot (k \cdot x)^a = b \cdot k^a \cdot x^a = k^a \cdot b \cdot x^a = k^a \cdot f(x)\,.$$

This means the value of the function is multiplied by k^a. ■

Example 2.7
In table 2.1, we see how the function $f(x) = 4x^3$ increases. The function is a power function with exponent $a = 3$. If x is multiplied by 2, y is multiplied by $2^3 = 8$.

Table 2.1: Growth of $f(x) = 4x^3$.

x	y
1	4
2	32
4	256
8	2048

Example 2.8
A power function $f(x) = b \cdot x^2$ has a graph that passes through the point $(3, 7)$. Here the value of b is unknown, but despite this it is still possible to calculate a second point that the graph passes through.

If x is multiplied by 4,[2] we get the new value $3 \cdot 4 = 12$. We then get the new value of the function by multiplying the old one (7) with 4^2 (since the exponent $a = 2$). This yields

$$7 \cdot 4^2 = 7 \cdot 16 = 112\,.$$

Thus the graph of the function f passes through $(12, 112)$.

[2] There is nothing special about the number 4; it could be any positive number.

Multiplying by a number corresponds to adding or subtracting a percentage. The number k in theorem 2.6 must therefore correspond to a growth factor. The growth factor k can be written as $1 + r_x$ where r_x is the growth rate of the independent variable x. The number k^a is then the growth factor of the dependent variable y, i.e.

$$k = 1 + r_x \qquad \text{and} \qquad k^a = 1 + r_y\,,$$

where r_y is the growth rate of y.

Theorem 2.9

If x and y are connected by the power function $f(x) = b \cdot x^a$, the growth rates r_x and r_y of x and y are connected by the equation

$$1 + r_y = (1 + r_x)^a .$$

Example 2.10
The function $f(x) = 4.2 \cdot x^{0.5}$ is an increasing function. If x grows by a rate of 80%, we have $r_x = 0.80$. We then have

$$1 + r_y = (1 + 0.80)^{0.5} = 1.342 .$$

r_y must therefore be 0.342 which is the same as 34.2%. Every time x grows by 80%, y grows by 34.2%.

Example 2.11
The function $f(x) = 5x^{-2}$ is a decreasing function with $a = -2$. If x grows by 40%, we have $r_x = 0.40$, i.e.

$$1 + r_y = (1 + 0.40)^{-2} = 0.510 .$$

This means

$$r_y = 0.510 - 1 = -0.490 = -49\% .$$

If x increases by 40% y *decreases* by 49%.

Example 2.12
If the function f is given by the formula $f(x) = 2x^3$, and we know that y has grown by 50%; then how much did x grow?

To find out, we insert $r_y = 0.50$ into the equation and get

$$1 + 0.50 = (1 + r_x)^3 .$$

We then solve the equation:

$$\begin{aligned}
1 + 0.50 = (1 + r_x)^3 &\quad \Leftrightarrow \\
\sqrt[3]{1.50} = 1 + r_x &\quad \Leftrightarrow \\
\sqrt[3]{1.50} - 1 = r_x &\quad \Leftrightarrow \\
0.145 = r_x . &
\end{aligned}$$

We now know that x grew by 14.5% for y to grow by 50%.

2.3 PROPORTIONALITY

Two variables y and x are called directly proportional when

$$y = c \cdot x$$

where c is a constant. If we call the constant b, we have the relation

$$y = b \cdot x = b \cdot x^1 ,$$

which is a power relation with $a = 1$.

Inverse proportionality is also a power relation. Two variables x and y are inversely proportional when $x \cdot y = c$ which can also be written as[3]

$$y = \frac{b}{x} = b \cdot \frac{1}{x} = b \cdot x^{-1} \,.$$

[3] Here we use that $\frac{1}{x^n} = x^{-n}$.

Inverse proportionality is a power relation with $a = -1$.

Theorem 2.13

For a power relation $y = b \cdot x^a$ we have

1. if $a = 1$, x and y are directly proportional.
2. if $a = -1$, x and y are inversely proportional.

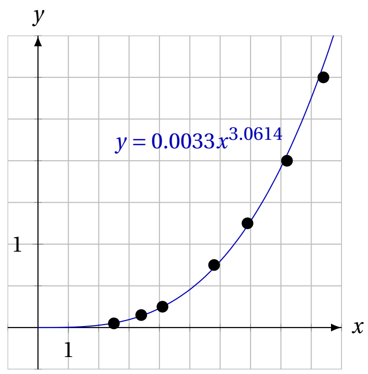


Figure 2.3: A power model can be found through power regression.

A direct proportionality can therefore be described by a power function

$$f(x) = b \cdot x \,.$$

The graph of this function is a straight line through $(0,0)$.[4] This means it is also a linear function with slope b intercepting the y-axis in 0.

[4] In principle, it is only a power function for positive values of x. But in this case there is no problem in letting x assume negative values.

Direct proportionality can therefore be seen as either a power function with exponent 1 or a linear function intercepting the y-axis in 0.

2.4 POWER REGRESSION

If a series of data points can be described by a power model, it is possible (as it is for linear and exponential models) to find an expression for the power function that best fits the data. This method is called power regression.[5]

[5] Since for a power function the independent and dependent variables are always positive, none of the data points may have coordinates that are negative or 0.

In figure 2.3, we see a series of data points. The power model in the figure is found using power regression on a computer. How this is done depends on which program is used.

Trigonometry

3

Trigonometry means "measurement of triangles". Therefore, before we begin the subject, we sum up some general results concerning triangles.

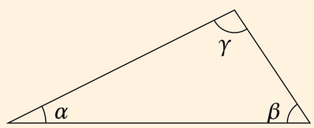
3.1 TRIANGLES

A triangle has three angles. For these we have a well known theorem which we do not prove.

Theorem 3.1

The sum of the angles of a triangle is 180°:

$$\alpha + \beta + \gamma = 180^\circ \,.$$



From each vertex in a triangle the following three lines may be drawn:

A median is a line from one vertex to the *midpoint* of the opposite side.

An angle bisector is a line from one vertex to the opposite side which cuts the corresponding *angle in half*.

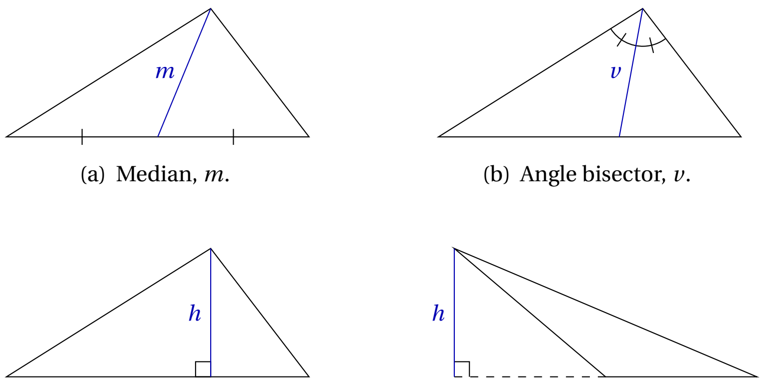
An altitude (or height) is a line from one vertex *perpendicular to* the opposite side. If the triangle is obtuse[1] the altitude lies outside the triangle.

[1] A triangle is called *obtuse* if one of its angles is obtuse, i.e. greater than 90°.

The three kinds of lines are shown in figure 3.1.

The altitude is perpendicular to the opposite side. This side is called the *base*, and it can be used with the altitude to calculate the area of a triangle.

We have the following well-known theorem, which connects the area, the altitude, and the base:



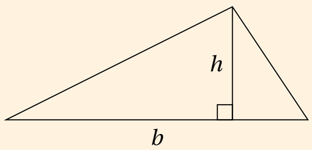
(a) Median, m. (b) Angle bisector, v. (c) Altitude, h. (d) Altitude outside the triangle.

Figure 3.1: Every vertex in a triangle has a corresponding median, angle bisector and altitude. Note that an altitude may lie outside the triangle.

Theorem 3.2

The area T of a triangle with altitude h and base b is given by the formula

$$T = \tfrac{1}{2} \cdot h \cdot b\,.$$



Notation

When discussing the sides and angles of a triangle, it is important to have an unambiguous notation. This ensures that it is clear at all times which side or which angle we are referring to. We follow the convention illustrated in figure 3.2: Vertices are denoted by a capital letter, and the sides are denoted by their end points. Angles are denoted by the same symbol as the corresponding vertex, i.e. $\angle A$ or just A for the angle located at A.

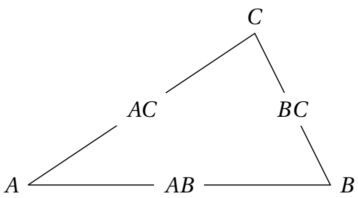


Figure 3.2: Sides and angles in $\triangle ABC$.

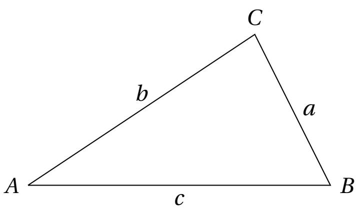


Figure 3.3: $\triangle ABC$ where the sides are called a, b and c.

Sometimes sides are denoted by the opposite angle. The side opposite angle A is then called a, etc. This can be seen in figure 3.3.

This notation (lowercase letters for names of the sides) is only useful if we look at just one triangle. If we have a more complicated figure with more than three vertices, more than one side can be said to lie opposite a certain angle. Using lowercase letters for sides will then be ambiguous. In this situation we have to call the sides AB, BC, etc.

If a figure has more than three vertices, ambiguity may also arise if we use only one letter to denote an angle, e.g. $\angle A$. In this case we denote an angle using 3 letters. The angle $\angle CAD$ can be found by drawing a line from C via A to D (see figure 3.4).

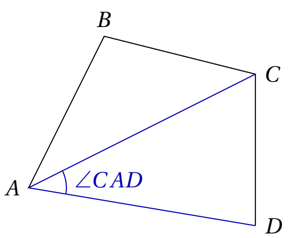


Figure 3.4: There are three different angles in this figure that could be called $\angle A$. The marked angle is therefore denoted by $\angle CAD$.

Similar triangles

Two triangles that have equal angles are called *similar*. We define similarity of triangles in the following way:

Definition 3.3

If for the triangles ABC og $A'B'C'$ we have

$$\angle A' = \angle A, \quad \angle B' = \angle B \quad \text{and} \quad \angle C' = \angle C\,,$$

the two triangles are called *similar*.

If two triangles are similar, one of them will be a smaller or larger copy of the other one. We have the following theorem:

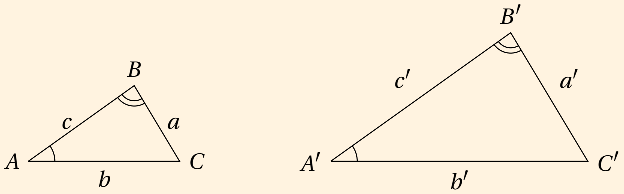
Theorem 3.4

If $\triangle ABC$ and $\triangle A'B'C'$ are similar with

$$\angle A' = \angle A, \quad \angle B' = \angle B \quad \text{and} \quad \angle C' = \angle C\,,$$

then the proportions between the corresponding[2] sides are equal, i.e.

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}\,.$$



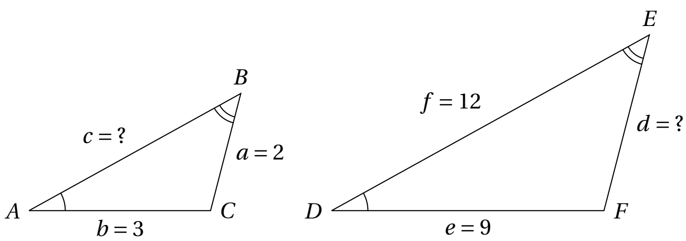
[2] We call two sides *corresponding* when they lie between equal angles.

Example 3.5
If $\triangle ABC$ and $\triangle DEF$ are similar and

$$\angle A = \angle D, \quad \angle B = \angle E, \quad \text{and} \quad \angle C = \angle F\,,$$

and we also know that $a = 2$, $b = 3$, $e = 9$, and $f = 12$, we can calculate the length of the remaining sides.

First, we draw a sketch (it does not have to be to scale) to get an overview:

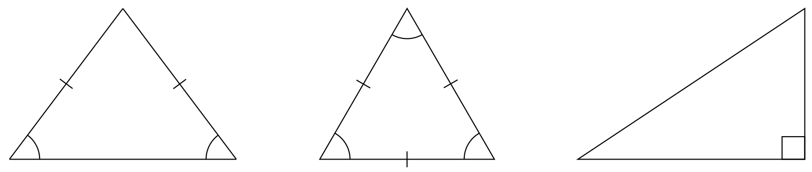


Then we look at the proportions between corresponding sides. In this case we have

$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c}\,.$$

When we insert the known values, we get

$$\frac{d}{2} = \frac{9}{3} = \frac{12}{c}\,.$$



(a) An isosceles triangle (b) An equilateral triangle. (c) A right-angled triangle.

Figure 3.5: An isosceles triangles has two sides of equal length, an equilateral triangle has three sides of equal length, and a right-angled triangle has a right angle.

Therefore

$$\frac{d}{2} = \frac{9}{3} \quad \Leftrightarrow \quad d = \frac{9}{3} \cdot 2 = 6\,,$$

and

$$\frac{9}{3} = \frac{12}{c} \quad \Leftrightarrow \quad \frac{c}{12} = \frac{3}{9} \quad \Leftrightarrow \quad c = \frac{3}{9} \cdot 12 = 4\,.$$

Now we know the length of every side in the two triangles.

Special triangles

There are three types of triangles, which have special names. These are *isosceles* triangles, *equilateral* triangles, and *right-angled* triangles. These can be seen in figure 3.5.

In right-angled triangles, the sides have names as well. The side opposite the right angle is called the *hypotenuse* and the two other sides are called *legs*[3] (cf. figure 3.6).

[3] They are also sometimes called *catheti*, singular *cathetus*.

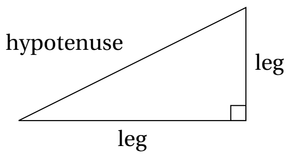


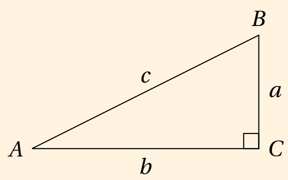
Figure 3.6: The sides of a right-angled triangle.

For right-angled triangles, the following well-known theorem holds:

Theorem 3.6: The Pythagorean theorem

For a right-angled triangle, ABC where $\angle C$ is the right angle, we have

$$a^2 + b^2 = c^2\,.$$



If we instead use the names of the sides of a right-angled triangle, we can write the Pythagorean theorem in this form:

Theorem 3.7: The Pythagorean theorem

In a right-angled triangle the sum of the squares of the legs is equal to the square of the hypotenuse, i.e.

$$(\text{first leg})^2 + (\text{second leg})^2 = (\text{hypotenuse})^2\,.$$

Example 3.8

If the length of one leg of a right-angled triangle is 5, and the length of the hypotenuse is 13, we can calculate the length of the last leg using the Pythagorean theorem.

We denote the length of the last leg by x. Inserting the known values into the formula yields the equation

$$5^2 + x^2 = 13^2 ,$$

i.e.

$$x^2 = 13^2 - 5^2 = 169 - 25 = 144 .$$

The length of the last leg is then

$$x = \sqrt{144} = 12.$$

3.2 SINE, COSINE AND TANGENT

If we want to calculate missing sides and angles in a triangle, it is necessary to have formulas for the correlation between the sides and angles of triangles. It is therefore useful to define the three *trigonometric* functions sine, cosine and tangent.

In figure 3.7, we see a coordinate system and a *unit circle*, i.e. a circle with its centre at $(0,0)$ and radius 1. The angle θ in the figure has one leg along the x-axis. The other leg intersects the circle at P. Angles are always drawn with one leg along the positive x-axis and the other leg corresponding to a counter-clockwise rotation (this is called the *positive* direction). If the angle is negative the rotation is clockwise.

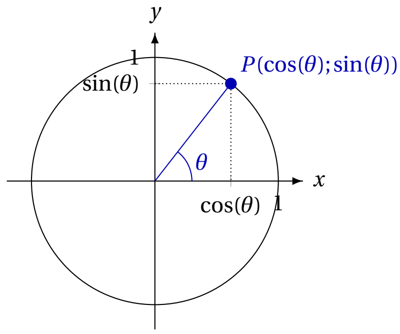


Figure 3.7: The sine and cosine of an angle can be found via the unit circle.

The three trigonometric functions are defined in the following way:

Definition 3.9

Let P be the point on the unit circle corresponding to the angle θ. Then

1. $\cos(\theta)$ is the x-coordinate of P.
2. $\sin(\theta)$ is the y-coordinate of P.
3. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

Note that $\tan(\theta)$ is only defined for those angles where $\cos(\theta) \neq 0$.

Using these functions, we are able to calculate angles from lengths and vice versa, since θ is an angle while $\cos(\theta)$ and $\sin(\theta)$ are coordinates, which are a sort of lengths.

Table 3.1: The values of $\cos(\theta)$ and $\sin(\theta)$ for certain angles.

θ	$\cos(\theta)$	$\sin(\theta)$
$20°$	0.940	0.342
$45°$	0.707	0.707
$60°$	0.5	0.866
$90°$	1	0
$100°$	-0.174	0.985

In table 3.1, the values of the cosine and the sine of certain angles are listed. From the table, we see that $\cos(60°) = 0.5$ and $\sin(60°) = 0.866$. This means that if the angle θ in figure 3.7 is $60°$, then the point P has coordinates $(0.5, 0.866)$.

From figure 3.7, we see that if the angle θ is between $90°$ and $180°$, then $\cos(\theta)$ is negative. This can be seen in the table as well.

Since $\cos(\theta)$ and $\sin(\theta)$ are the coordinates of a point on the unit circle, and the unit circle has radius 1, it follows that the values of $\sin(\theta)$ and $\cos(\theta)$ must be numbers between -1 and 1:

$$-1 \leq \cos(\theta) \leq 1 \qquad \text{and} \qquad -1 \leq \sin(\theta) \leq 1 .$$

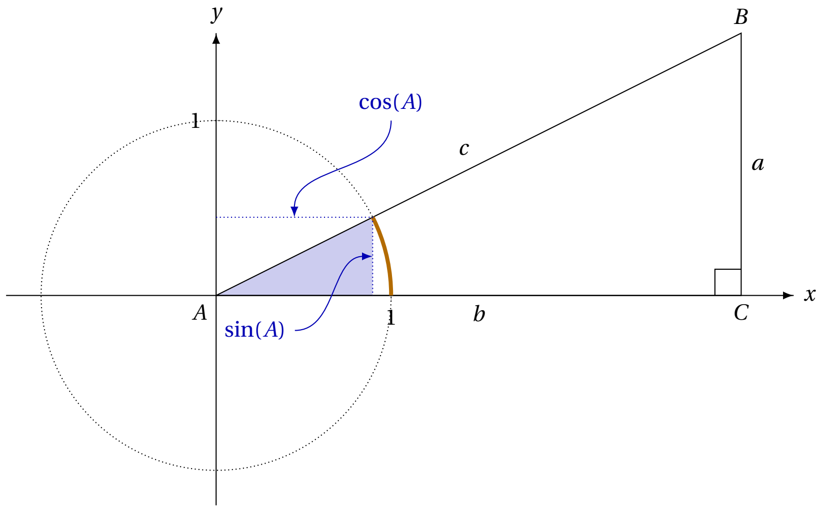


Figure 3.8: A right-angled triangle where the angle A is placed at the origin of a coordinate system. $\sin(A)$ and $\cos(A)$ are lengths of sides in the small marked triangle which is similar to the triangle ABC.

However, as long as we look only at triangles, an angle θ is always less than 180°. In this case, $\sin(\theta)$ is always positive.

Using the reflectional and the rotational symmetry of the unit circle, it is possible to deduce certain properties of the sine and the cosine, as well as relations between them. We have the following theorem, which we do not prove:

Theorem 3.10

If θ is any angle then

1. $\cos(90° - \theta) = \sin(\theta)$
2. $\sin(90° - \theta) = \cos(\theta)$
3. $\cos(-\theta) = \cos(\theta)$
4. $\sin(-\theta) = -\sin(\theta)$
5. $\cos(180° - \theta) = -\cos(\theta)$
6. $\sin(180° - \theta) = \sin(\theta)$

Since the radius of the unit circle is 1, we also have the following relation between the cosine and the sine. It can be derived from the Pythagorean theorem.

Theorem 3.11

Let θ be any angle. Then

$$\cos(\theta)^2 + \sin(\theta)^2 = 1\,.$$

3.3 TRIGONOMETRY IN RIGHT-ANGLED TRIANGLES

First, we look at the use of trigonometric functions in right-angled triangles.

In figure 3.8, we see a right-angled triangle placed in a coordinate system. A unit circle is also drawn. The angle A is an angle in the unit circle as well as an angle in the triangle ABC.

Now, $\cos(A)$ and $\sin(A)$ can be found as lengths of sides in the small, marked triangle. They are the horizontal and vertical distances marked in figure 3.8 (see definition 3.9). $\tan(A)$ is the slope of the line AB in the figure.

The marked triangle is similar to $\triangle ABC$. The hypotenuse of this small triangle is 1 since it is equal to the radius of the circle. Because the hypotenuse of triangle ABC is c, we can calculate the lengths of the sides of the small triangle by dividing the lengths of the sides in triangle ABC by c.[4]

[4]The two triangles are similar, and their hypotenuses are 1 and c. Thus $\triangle ABC$ is c times larger than the small triangle, and the small triangle is c times smaller than $\triangle ABC$.

The length of the vertical leg of the small triangle is $\sin(A)$, and the length of the vertical leg of $\triangle ABC$ is a. Therefore we know that

$$\sin(A) = \frac{a}{c}\,.$$

In the small triangle, the length of the horizontal leg is $\cos(A)$, and in $\triangle ABC$, the length of the horizontal leg is b. Thus

$$\cos(A) = \frac{b}{c}\,.$$

From these two results, we can obtain a formula for $\tan(A)$, since

$$\tan(A) = \frac{\sin(A)}{\cos(A)} = \frac{a}{c}\Big/\frac{b}{c} = \frac{a}{c}\cdot\frac{c}{b} = \frac{a}{b}\,.$$

The calculations above are summed up in the following theorem:

Theorem 3.12

In a right-angled triangle ABC where C is the right angle

$$\sin(A) = \frac{a}{c}\,, \quad \cos(A) = \frac{b}{c} \quad \text{and} \quad \tan(A) = \frac{a}{b}\,.$$

Not all triangles are called ABC. Therefore we sometimes write theorem 3.12 in the following way:

Theorem 3.13

In a right-angled triangle, where θ is one of the acute angles,

$$\sin(\theta) = \frac{\text{opposite leg}}{\text{hypotenuse}}\,,$$
$$\cos(\theta) = \frac{\text{adjacent leg}}{\text{hypotenuse}}\,,$$
$$\tan(\theta) = \frac{\text{opposite leg}}{\text{adjacent leg}}\,.$$

The terms “opposite” and “adjacent” leg refer to the placement of the leg in relation to the angle θ. See figure 3.9.

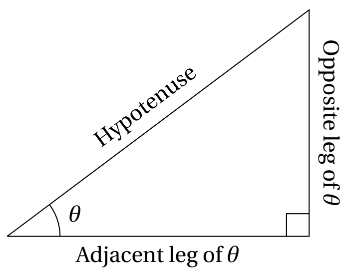
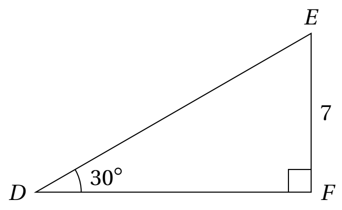


Figure 3.9: The names of the sides in a right-angled triangle in relation to the angle θ.

Using the formulas in theorem 3.13, we can calculate every side and angle of a right-angled triangle if we know at least one side and one of the acute angles, or another one of the sides. This is because the value of e.g. $\sin(32°)$ or $\cos(51°)$ can be found using a calculator.

Example 3.14
In a right-angled triangle DEF, $\angle D = 30°$ and $|EF| = 7$. A sketch of the triangle might look like this:



Since EF is the opposite leg of $\angle D$, and DE is the hypotenuse, according to theorem 3.13 we have

$$\sin(30°) = \frac{7}{|DE|} .$$

We solve this equation and get[5]

$$|DE| = \frac{7}{\sin(30°)} = 14 .$$

[5] The result can be calculated using a calculator.

Using the Pythagorean theorem, we can determine the length of the last side, and because the sum of the angles is 180°, we can determine the last angle.

Example 3.15
If we want to determine the length of the leg DF of the triangle from example 3.14, we can use the tangent function.

The side DF is the adjacent leg of $\angle D$, and EF is the opposite. From theorem 3.13, we get

$$\tan(30°) = \frac{7}{|DF|} .$$

If we solve this equation, we get

$$|DF| = \frac{7}{\tan(30°)} = 12.1 .$$

3.4 INVERSE TRIGONOMETRIC FUNCTIONS

In section 3.3, we described how to calculate the sides of a right-angled triangle from the sine, cosine and tangent. In this section we describe how to find an *angle* if its sine, cosine or tangent is known.

For this calculation we use the *inverse trigonometric functions* $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.[6] The three functions are used to solve equations where we know the sine, cosine or tangent of an unknown angle.

[6] The three functions are sometimes denoted by arccos, arcsin and arctan because they give us the "arc", i.e. angle, which has a certain sine, cosine or tangent value.

In computer programs the three functions are often denoted by asin, acos and atan.

Example 3.16
To solve the equation $\cos(v) = 0.8$, we use $\cos^{-1}$:

$$\cos(v) = 0.8 \qquad \Leftrightarrow \qquad v = \cos^{-1}(0.8) .$$

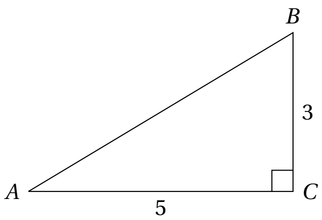
We find $\cos^{-1}(0.8)$ using a calculator and get

$$v = \cos^{-1}(0.8) = 36.9° .$$

Example 3.17
The equation $\sin(B) = 0.5$ is solved in the following way:

$$\sin(B) = 0.5 \qquad \Leftrightarrow \qquad B = \sin^{-1}(0.5) = 30^\circ .$$

Example 3.18
In a right-angled triangle ABC, we have $|AC| = 5$ and $|BC| = 3$. A sketch of the triangle looks like this:



The side AC is the adjacent leg of $\angle A$, and the side BC is the opposite leg. According to theorem 3.13, we have

$$\tan(A) = \frac{3}{5} = 0.6 .$$

The solution to the equation $\tan(0.6)$ can be found using $\tan^{-1}$. We get

$$\tan(A) = 0.6 \qquad \Leftrightarrow \qquad A = \tan^{-1}(0.6) = 30.96^\circ .$$

The last angle can be found from the sum of the angles, and the last side (the hypotenuse) can be found using the Pythagorean theorem.

3.5 THE AREA OF A TRIANGLE

In the preceding sections we looked only at right-angled triangles. However, most triangles are not right-angled, therefore we need to find formulas that apply in the general case.

It turns out that we can derive formulas for the general case by drawing an altitude, thereby dividing the triangle into two right-angled triangles. We can then use the formulas we already know for right-angled triangles to prove more general formulas, e.g. this theorem:

Theorem 3.19

The area T of a triangle ABC is

$$T = \tfrac{1}{2} \cdot a \cdot b \cdot \sin(C) ,$$
$$T = \tfrac{1}{2} \cdot a \cdot c \cdot \sin(B) ,$$
$$T = \tfrac{1}{2} \cdot b \cdot c \cdot \sin(A) .$$

Proof
We first assume that the angle C is acute. In $\triangle ABC$ we draw the altitude h_B from B (cf. figure 3.10). According to theorem 3.2 the area of $\triangle ABC$ is

$$T = \tfrac{1}{2} \cdot h_B \cdot b. \tag{3.1}$$

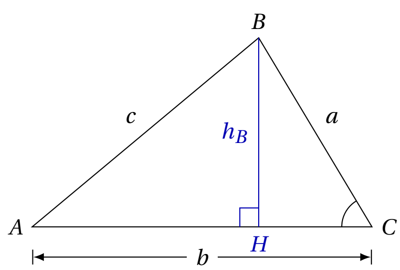


Figure 3.10: Triangle ABC with altitude h_B. The angle C is acute.

But $\triangle BCH$ is right-angled so we have (theorem 3.13)

$$\sin(C) = \frac{h_B}{a} \qquad \Leftrightarrow \qquad h_B = a \cdot \sin(C)\,.$$

We insert this expression for h_B into the equation (3.1) and get

$$T = \tfrac{1}{2} \cdot a \cdot \sin(C) \cdot b = \tfrac{1}{2} \cdot a \cdot b \cdot \sin(C)\,.$$

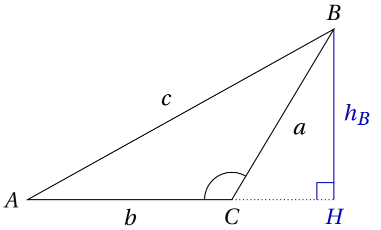


Figure 3.11: Triangle ABC with altitude h_B. The angle C is obtuse.

If the angle C is obtuse, the triangle looks like the one in figure 3.11. Here we have

$$\angle BCH = 180^\circ - \angle C\,.$$

According to theorem 3.10, this means that

$$\sin(\angle BCH) = \sin(180^\circ - C) = \sin(C)\,.$$

If we look at the right-angled $\triangle BCH$, we find that

$$\sin(\angle BCH) = \frac{h_B}{a}\,,$$

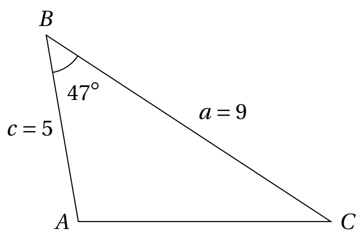
i.e.

$$\sin(C) = \frac{h_B}{a}\,.$$

Thus we end up with the same formula as in the acute case. ■

This only proves one of the formulas in theorem 3.19. But if we look closely at the formulas, we see that in every formula, the area is calculated from two sides and the *enclosed* angle. Therefore it is not necessary to prove all the three formulas individually—since it is actually the same formula written in three different ways.

Example 3.20
In $\triangle ABC$, we have $\angle B = 47^\circ$, $a = 9$ and $c = 5$. A sketch of the triangle looks like this:



The area of this triangle is

$$T = \tfrac{1}{2} \cdot a \cdot c \cdot \sin(B) = \tfrac{1}{2} \cdot 9 \cdot 5 \cdot \sin(47^\circ) = 16.5\,.$$

Example 3.21
In $\triangle DEF$, we have $d = 4$ and $\angle E = 75^\circ$. If we know that the area is 13, we can calculate the length of the side f and draw a sketch of the triangle.

We first notice that the adjacent sides of $\angle E$ are the sides d and f. Therefore theorem 3.19 gives the equation

$$T = \tfrac{1}{2} \cdot d \cdot f \cdot \sin(E)\,.$$

If we insert the known values into the formula, we get the equation

$$13 = \tfrac{1}{2} \cdot 4 \cdot f \cdot \sin(75°)\,.$$

This equation has the solution

$$f = \frac{2 \cdot 13}{4 \cdot \sin(75°)} = 6.7\,.$$

Thus the length of the side f is 6.7.

3.6 THE LAW OF SINES

From theorem 3.19, we can derive the following theorem:

Theorem 3.22: The law of sines

In a triangle ABC, we have

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}\,,$$

and

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}\,.$$

Proof

The three formulas in theorem 3.19 are formulas for the area of the same triangle. Therefore we must have

$$\tfrac{1}{2} \cdot b \cdot c \cdot \sin(A) = \tfrac{1}{2} \cdot a \cdot c \cdot \sin(B) = \tfrac{1}{2} \cdot a \cdot b \cdot \sin(C)\,.$$

In this double equation, we divide by $\frac{1}{2} \cdot a \cdot b \cdot c$ on all "sides" which yields

$$\frac{\frac{1}{2} \cdot b \cdot c \cdot \sin(A)}{\frac{1}{2} \cdot a \cdot b \cdot c} = \frac{\frac{1}{2} \cdot a \cdot c \cdot \sin(B)}{\frac{1}{2} \cdot a \cdot b \cdot c} = \frac{\frac{1}{2} \cdot a \cdot b \cdot \sin(C)}{\frac{1}{2} \cdot a \cdot b \cdot c}\,.$$

Now we reduce as much as possible and get

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}\,,$$

which proves the theorem ■

The law of sines states that the proportion between the sine of an angle and its opposite side is constant in any given triangle. If we know an angle and an opposite side, we can calculate this constant. Then, if we know either another angle or another side, we can calculate the rest of the angles and sides of the triangle.

Example 3.23
In $\triangle ABC$, we have $\angle C = 47°$, $a = 5$ and $c = 8$. A sketch can be seen in figure 3.12.

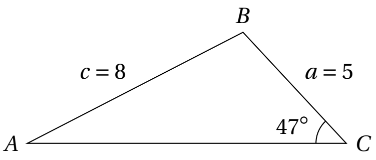


Figure 3.12: A triangle with two known sides and one known angle.

According to the law of sines (theorem 3.22)

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c},$$

i.e.

$$\frac{\sin(A)}{5} = \frac{\sin(47°)}{8} \quad \Leftrightarrow \quad \sin(A) = \frac{\sin(47°)}{8} \cdot 5 = 0.4571\,.$$

Therefore

$$\angle A = \sin^{-1}(0.4571) = 27.2°\,.$$

The angle B kan be found from the sum of the angles, and the last side can then also be calculated using the law of sines (see the next example).

Example 3.24
In $\triangle ABC$, we have $\angle A = 62°$, $\angle B = 34°$ and $b = 7$. A sketch of the triangle can be found in figure 3.13.

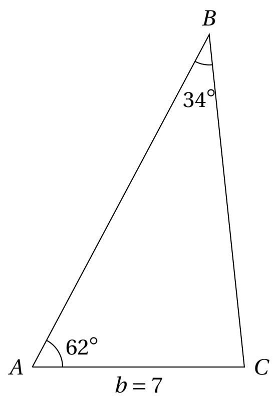


Figure 3.13: A triangle with two known angles and one known side.

According to the law of sines

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)},$$

i.e.

$$\frac{a}{\sin(62°)} = \frac{7}{\sin(34°)} \quad \Leftrightarrow \quad a = \frac{7}{\sin(34°)} \cdot \sin(62°) = 11.1\,.$$

The last angle can be calculated from the sum of all the angles, and the last side can be found in the same way as a.

The Ambiguous Case

It turns out that we have to be careful when using the law of sines to calculate angles. The function $\sin^{-1}$ which we use to calculate angles, always returns a value between 0 and 90°; but in a triangle an angle can be up to 180°—and it turns out that for a given sine value, two different angles might yield this value.

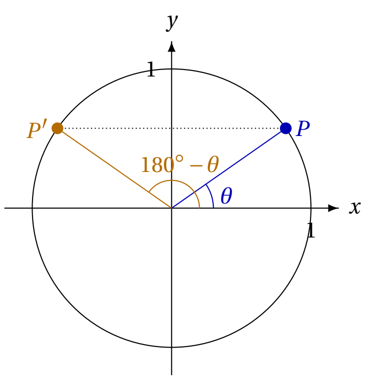


Figure 3.14: Two different angles with the same sine.

If we look at figure 3.14, we see that the angle θ and the angle $180° - \theta$ have corresponding points on the unit circle (P and P') with the same y-coordinate, i.e.

$$\sin(180° - \theta) = \sin(\theta)\,.$$

From this we find that the equation $\sin(\theta) = y$ might have two solutions. These solutions are

$$\theta = \sin^{-1}(y) \qquad \text{and} \qquad \theta = 180° - \sin^{-1}(y)\,.$$

Example 3.25
In the triangle ABC, we have $\angle A = 56°$, $a = 7$ and $b = 8$. Using the law of sines, we can calculate $\angle B$ because

$$\frac{\sin(B)}{b} = \frac{\sin(A)}{a} .$$

We now have the equation

$$\frac{\sin(B)}{8} = \frac{\sin(56°)}{7} \qquad \Leftrightarrow \qquad \sin(B) = \frac{\sin(56°)}{7} \cdot 8 = 0.9475 .$$

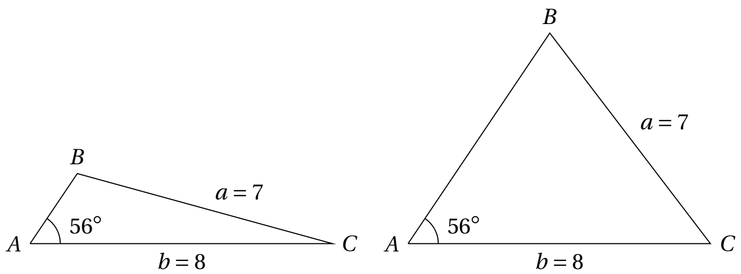
This equation has two solutions. The first solution is[7]

$$\angle B = \sin^{-1}(0.9475) = 71,3° ,$$

and the other one is

$$\angle B = 180° - \sin^{-1}(0.9475) = 108.7° .$$

Therefore the triangle ABC can be two different triangles:



If we want to calculate the remaining sides and angles of $\triangle ABC$, we need to do the calculations for both of these triangles. Therefore there is not one but two solutions, and both of them are equally correct.

Even though the equation $\sin(\theta) = y$ always has two solutions, both solutions do not necessarily make sense. This can be seen in the next example:

Example 3.26
Here we look a the triangle ABC where $\angle A = 45°$, $a = 15$ and $b = 12$. The law of sines yields

$$\frac{\sin(B)}{b} = \frac{\sin(A)}{a} ,$$

whence we get the equation

$$\frac{\sin(B)}{12} = \frac{\sin(45°)}{15} \qquad \Leftrightarrow \qquad \sin(B) = \frac{\sin(45°)}{15} \cdot 12 = 0.5657 .$$

This equation has two solutions:

$$\angle B = \sin^{-1}(0.5657) = 34.4°$$

and

$$\angle B = 180° - \sin^{-1}(0.5657) = 145.6° .$$

[7] There are two solutions to $\sin(B) = 0.9475$ because $\sin(71,3°) = \sin(108,7°)$, which means both these angles solve the equation.

The last solution ($\angle B = 145.6°$) is a solution to the equation $\sin(B) = 0.5657$ but it is not a meaningful solution. The sum of the angles in a triangle is 180°, and we already have an angle of 45°. There is no room for an angle of 145.6° in the triangle, and we therefore reject this solution.

Thus angle B can have only one possible value, namely 34.4°.

3.7 THE LAW OF COSINES

The law of sines can be used whenever we know an angle and an opposite side. In the cases where we do not know an angle and an opposite side, we instead need to use the *law of cosines*.

Theorem 3.27: The law of cosines

In a triangle ABC, we have

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(A) \qquad \cos(A) = \frac{b^2 + c^2 - a^2}{2 \cdot b \cdot c}$$
$$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos(B) \qquad \cos(B) = \frac{a^2 + c^2 - b^2}{2 \cdot a \cdot c}$$
$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(C) \qquad \cos(C) = \frac{a^2 + b^2 - c^2}{2 \cdot a \cdot b} .$$

The formulas on the right hand side in theorem 3.27 are the same as those on the left hand side—but with a different emphasis.

If we take a closer look at the formulas on the left hand side, we will see that all three formulas enable us to calculate a side if we know the opposite angle and the other two sides. Since this is true for all three formulas, we only need to prove one of them.

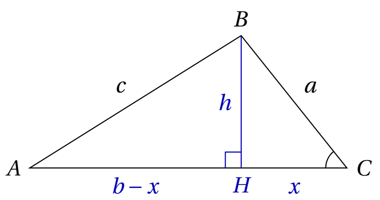


Figure 3.15: Triangle ABC with altitude h. The angle C is acute.

Proof

We first assume that the angle C is acute. In the triangle ABC, we draw the altitude h from B. The point where the altitude intersects the base, we call H (cf. figure 3.15).

If we use the Pythagorean theorem on the right-angled triangles ABH and BCH, we get

$$c^2 = h^2 + (b-x)^2 \qquad \text{and} \qquad a^2 = h^2 + x^2 .$$

In both of these equations we isolate h^2 to obtain

$$c^2 - (b-x)^2 = h^2 \qquad \text{and} \qquad a^2 - x^2 = h^2 .$$

Now we have two expressions that equal h^2. These two expressions must therefore also be equal, i.e.

$$\begin{aligned} c^2 - (b-x)^2 &= a^2 - x^2 &&\Leftrightarrow \\ c^2 &= a^2 - x^2 + (b-x)^2 &&\Leftrightarrow \\ c^2 &= a^2 - x^2 + b^2 + x^2 - 2 \cdot b \cdot x &&\Leftrightarrow \end{aligned}$$

$$c^2 = a^2 + b^2 - 2 \cdot b \cdot x \,. \tag{3.2}$$

Since the triangle BCH is right-angled, theorem 3.13 implies

$$\cos(C) = \frac{x}{a} \qquad \Leftrightarrow \qquad x = a \cdot \cos(C)\,.$$

We insert this result into equation (3.2) and get

$$c^2 = a^2 + b^2 - 2 \cdot b \cdot a \cdot \cos(C)\,.$$

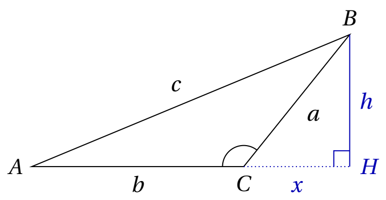


Figure 3.16: Triangle ABC with altitude h. The angle C is obtuse.

If angle C is obtuse, the altitude lies outside the triangle, see figure 3.16. Here we use the Pythagorean theorem on the two right-angled triangles ABH and BCH to get

$$c^2 = h^2 + (b+x)^2 \qquad \text{and} \qquad a^2 = h^2 + x^2\,.$$

If we manipulate these two equations in the same way as we did in the acute case, we get

$$c^2 = a^2 + b^2 + 2 \cdot b \cdot x\,, \tag{3.3}$$

which differs from (3.2) in the sign of the last term.

Now, we look at triangle BCH and find that

$$\cos(\angle BCH) = \frac{h}{a} \qquad \Leftrightarrow \qquad h = a \cdot \cos(\angle BCH)\,.$$

But $\angle BCH = 180° - \angle C$, so theorem 3.10 yields

$$\cos(\angle BCH) = \cos(180° - C) = -\cos(C)\,,$$

i.e.

$$h = a \cdot \cos(\angle BCH) = -a \cdot \cos(C)\,.$$

When we insert this into equation (3.3), we again obtain the formula

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(C)\,. \qquad \blacksquare$$

The law of cosines can be used to calculate a side if we know the other two sides of a triangle as well as the opposite angle. To do this, we use the formulas on the left hand side in theorem 3.27.

We can also use the law of cosines to calculate an angle if we know all three sides of a triangle—here we use the formulas on the right hand side of the theorem.

Example 3.28

In triangle ABC, we have $\angle C = 39°$, $a = 7$ and $b = 10$. A sketch of the triangle can be seen in figure 3.17.

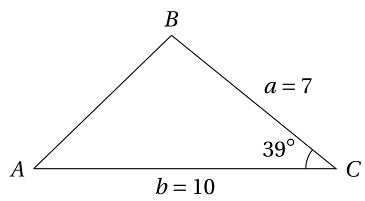


Figure 3.17: Triangle where two sides and the enclosed angle is known.

The side c can be calculated from the law of cosines. According to theorem 3.27

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(C) = 7^2 + 10^2 - 2 \cdot 7 \cdot 10 \cdot \cos(39°) = 40.20\,,$$

which means that

$$c = \sqrt{40.20} = 6.3\,.$$

Now we know every side of the triangle. We can then calculate one of the last two angles using the law of cosines (see next example). Alternatively, we can find one of the last two angles using the law of sines.

Example 3.29

In this example, we look at triangle ABC where $a = 3$, $b = 6$ and $c = 4$ (see figure 3.18). According to the law of cosines, $\angle A$ can be found using the formula

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{6^2 + 4^2 - 3^2}{2 \cdot 6 \cdot 4} = 0.8958\,.$$

From this we get

$$\angle A = \cos^{-1}(0.9858) = 26.4^\circ\,.$$

The remaining angles can be calculated in the same way.

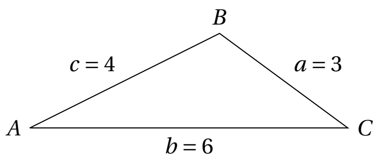


Figure 3.18: Triangle where all three sides are known.

Polynomials 4

The function

$$f(x) = 3x^2 + x - 4$$

belongs to the group of *polynomials*, which are a type of function used in many different branches of mathematics. This is because, as it turns out, polynomials are very well-behaved functions. We define polynomials in the following way:

Definition 4.1

A *polynomial* is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the coefficients $a_0, \ldots, a_n$ are real numbers, and $a_n \neq 0$.

The number n, which is a whole number, is called the *degree* of the polynomial.

Some examples of polynomials are

$$\begin{aligned}
&\textit{1st degree} \text{ polynomial:} && f(x) = 3x + 1 \\
&\textit{2nd degree} \text{ polynomial:} && g(x) = 4x^2 - 3x + 5 \\
&\textit{3rd degree} \text{ polynomial:} && h(x) = x^3 + 7x - 13 \\
&\textit{4th degree} \text{ polynomial:} && m(x) = 8x^4 + 7x^2 \\
&\textit{17th degree} \text{ polynomial:} && p(x) = x^{17} + 4x^9 \,.
\end{aligned}$$

When we write the formula of a polynomial, we normally sort the terms, so that they are listed with the highest exponent first. This is not strictly necessary, but it does make it easier to find the degree of the polynomial—since the degree is the highest occuring exponent.[1]

[1] A special case are polynomials of degree 0 which are constant functions; e.g. $f(x) = 9$ or $g(x) = -14$.

Example 4.2
The polynomial $f(x) = x + 4 - 3x^2$ can be written as

$$f(x) = -3x^2 + x + 4\,,$$

which makes it easier to see that this is a second degree polynomial.

First degree polyomials are actually linear functions. They are therefore not treated in this chapter. Most of this chapter will instead be devoted to the properties of second degree polynomials—with a concluding section concerning polynomials of a higher degree.

4.1 SHIFTED GRAPHS

In this section, we describe a general method for shifting graphs horizontally and vertically. The method we describe, applies to all functions—not just polynomials.

In figure 4.1, we see how a graph can be shifted along the x- or the y-axis. If the graph of the function $f(x)$ is shifted along the x-axis by an amount x_0, we get the graph of a new function $g(x)$ where

$$g(x + x_0) = f(x)\,.$$

This can be rewritten as

$$g(x) = f(x - x_0)\,,$$

and we can use this to find a formula for g when we know a formula for f.

If we shift the graph of f along the y-axis by an amount y_0, we get the graph of the new function g where

$$g(x) = f(x) + y_0\,.$$

Shifting a graph in both directions simultaneously—by (x_0, y_0)—gives us the graph of a new function g whose formula can be found by combining the two results above. We therefore have the following theorem:

Theorem 4.3

If the graph of the function $f(x)$ is shifted by an amount $(x_0; y_0)$, we get the graph of the function

$$g(x) = f(x - x_0) + y_0\,.$$

Example 4.4
In figure 4.2, we see the graph of $f(x) = \sqrt{x}$ shifted by $(1,3)$. According to theorem 4.3 this is the graph of

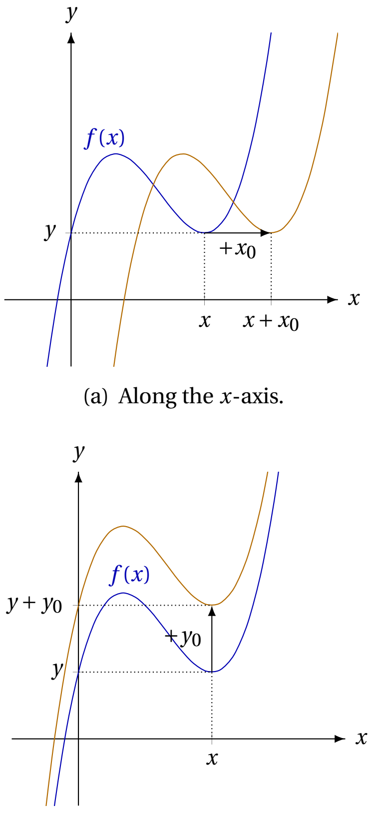
$$g(x) = f(x-1) + 3 = \sqrt{x-1} + 3\,.$$

4.2 SECOND DEGREE POLYNOMIALS

A *second degree polynomial* is, according to definition 4.1, a function of the form

$$f(x) = ax^2 + bx + c\,, \tag{4.1}$$

where a, b and c are three numbers, and $a \neq 0$.[2]



(a) Along the x-axis.

(b) Along the y-axis.

Figure 4.1: A graph can be shifted horizontally along the x-axis and vertically along the y-axis.

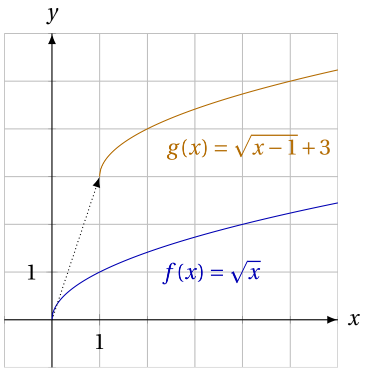


Figure 4.2: The graph of $f(x) = \sqrt{x}$ shifted by $(1,3)$.

[2]In the definition, the three coefficients are denoted by a_2, a_1 and a_0; but since a second degree polynomial only has three coefficients, we denote them by a, b and c.

The simplest second degree polynomial is a polynomial where the coefficients b and c are both 0, i.e.

$$p(x) = ax^2 .$$

The graph of $p(x) = ax^2$ can be seen in figure 4.3. This type of graph is called a parabola. As we see in the figure, the shape of the graph is given by the sign of a: If $a > 0$ the parabola *opens upward*; if $a < 0$ the parabola *opens downward*.

The reason for this is that x^2 is always a positive number. The sign of the function value thus depends on the sign of a.

From the figure, we also see that the parabola is symmetric, and that the axis of symmetry is the y-axis. This is because $(-x)^2 = x^2$, i.e. the polynomial $p(x)$ will have the same function value for x and $-x$.

Finally, we see in the figure that the highest or lowest point on the parabola is at $(0,0)$. This point is called the *vertex* of the parabola.

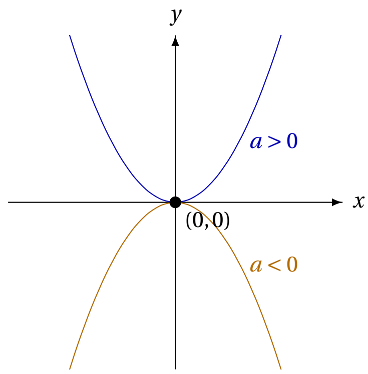


Figure 4.3: The graph of $p(x) = ax^2$ where $a > 0$ or $a < 0$.

If we instead want the vertex of the parabola to be placed in (x_0, y_0), we can shift the graph of $p(x)$. This is seen in figure 4.4. The new parabola will have the line $x = x_0$ as the axis of symmetry.

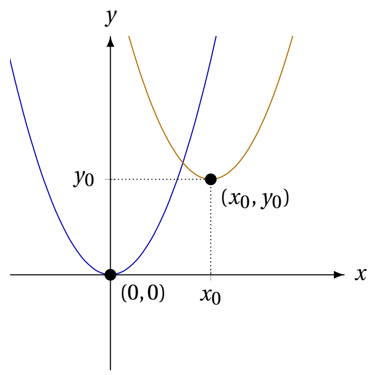


Figure 4.4: The graph of $p(x) = ax^2$ shifted to $f(x) = a(x - x_0)^2 + y_0$.

From theorem 4.3 we derive

Theorem 4.5

The parabola with vertex (x_0, y_0) is the graph of the function

$$f(x) = a(x - x_0)^2 + y_0 .$$

Proof

The parabola with vertex (x_0, y_0) can be obtained by shifting the graph of the parabola with vertex $(0,0)$ by (x_0, y_0).

The parabola with vertex $(0,0)$ is the graph of the function $p(x) = ax^2$. According to theorem 4.3, the parabola with vertex (x_0, y_0) is therefore the graph of the function

$$f(x) = p(x - x_0) + y_0 = a(x - x_0)^2 + y_0 . \qquad \blacksquare$$

The formula in theorem 4.5 does not look like the second degree polynomial in (4.1). However, it is the same function, and we can rewrite one of the formulas to look exactly like the other.

Example 4.6

The graph of $f(x) = 3 \cdot (x-2)^2 - 7$ is a parabola with vertex $(2, -7)$.

The formula for the function can be rewritten like this:

$$\begin{aligned} f(x) &= 3(x-2)^2 - 7 \\ &= 3(x^2 + (-2)^2 - 2 \cdot 2 \cdot x) - 7 \\ &= 3(x^2 + 4 - 4x) - 7 \\ &= 3x^2 + 12 - 12x - 7 \end{aligned}$$

$$= 3x^2 - 12x + 5\,.$$

This means the formula $f(x) = 3(x-2)^2 - 7$ can be written as

$$f(x) = 3x^2 - 12x + 5\,,$$

which is exactly like the form presented in (4.1) with coefficients

$$a = 3, \quad b = -12 \quad \text{and} \quad c = 5\,.$$

The calculation in example 4.6 can be done generally. If we look at the second degree polynomial $f(x) = a(x - x_0)^2 + y_0$, we get

$$\begin{aligned} f(x) &= a(x - x_0)^2 + y_0 \\ &= a(x^2 + x_0^2 - 2x_0 x) + y_0 \\ &= ax^2 + ax_0^2 - 2ax_0 x + y_0 \\ &= ax^2 + (-2ax_0)x + (ax_0^2 + y_0)\,. \end{aligned}$$

If this corresponds to the formula

$$f(x) = ax^2 + bx + c\,,$$

the coefficients must be equal. From this we get

$$b = -2ax_0 \quad \text{and} \quad c = ax_0^2 + y_0\,. \tag{4.2}$$

The equations (4.2) can be used to determine the coefficients b and c when we know the vertex (x_0, y_0). Usually, the second degree polynomial will have the form (4.1), and we instead wish to determine the vertex when we know the three coefficients a, b and c.

To simplify the formula for the vertex, we introduce the *discriminant*,[3]

[3] The discriminant is used for calculating more than just the vertex. It turns up again in section 4.3 below.

$$d = b^2 - 4ac\,.$$

We then have the following theorem:

Theorem 4.7

The second degree polynomial $f(x) = ax^2 + bx + c$ has its vertex at (x_0, y_0) where

$$x_0 = -\frac{b}{2a} \quad \text{and} \quad y_0 = -\frac{d}{4a}\,.$$

$d = b^2 - 4ac$ is the *discriminant*.

Proof

To prove the theorem, we look at the equation (4.2). Here we have

$$b = -2ax_0 \qquad \Leftrightarrow \qquad -\frac{b}{2a} = x_0\,.$$

This proves the formula for x_0.

Since $c = ax_0^2 + y_0$, we have

$$y_0 = c - ax_0^2 .$$

We have just shown that $x_0 = -\frac{b}{2a}$ which implies

$$\begin{aligned} y_0 &= c - a\left(-\frac{b}{2a}\right)^2 = c - a \cdot \frac{b^2}{4a^2} = c - \frac{b^2}{4a} \\ &= \frac{4ac}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} = -\frac{b^2 - 4ac}{4a} = -\frac{d}{4a} . \end{aligned}$$

Now, we have also proven the formula for y_0. ■

Example 4.8
The graph of the second degree polynomial

$$f(x) = x^2 - 4x + 1$$

can be seen in figure 4.5. To calculate the vertex of this parabola, we first find the coefficients of the polynomial. They are

$$a = 1 , \quad b = -4 \quad \text{and} \quad c = 1 .$$

Now we can calculate the x-coordinate of the vertex:

$$x_0 = -\frac{b}{2a} = -\frac{-4}{2 \cdot 1} = 2 .$$

To calculate the y-coordinate, we first calculate the discriminant

$$d = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 1 = 12 .$$

Therefore the y-coordinate is

$$y_0 = -\frac{d}{4a} = -\frac{12}{4 \cdot 1} = -3 .$$

As we see in the figure, the parabola's vertex is at $(2, -3)$.

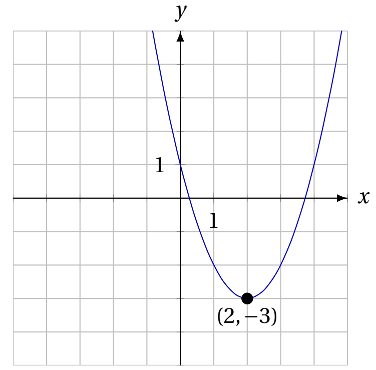


Figure 4.5: The graph of $f(x) = x^2 - 4x + 1$ has its vertex in $(2, -3)$.

4.3 QUADRATIC EQUATIONS

A parabola may intercept the x-axis. The values of x where a parabola intersects the x-axis are known as the *roots* of the polynomial.

Second degree polynomials may have two, one or no roots, depending on whether the parabola intercepts the x-axis twice, once or not at all (cf. figure 4.6).

To find the roots, we must find out where the parabola intercepts the x-axis. In every point on the x-axis, we have $y = 0$, i.e. the roots can be found where $f(x) = 0$. Thus we find the roots by solving the *quadratic equation*

$$ax^2 + bx + c = 0 .$$

In this equation, it is not readily seen how we may solve for x. Luckily, we have a solution formula. This is known as the *quadratic formula*:

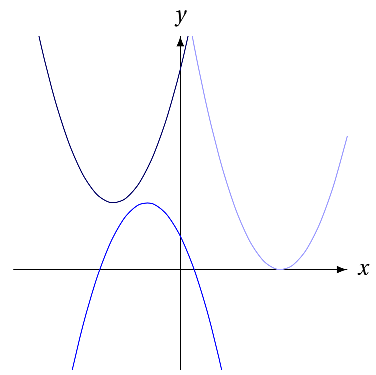


Figure 4.6: Parabolas may intercept the x-axis twice, once or not at all.

Theorem 4.9: The quadratic formula

To solve the quadratic equation

$$ax^2 + bx + c = 0$$

we calculate the discriminant $d = b^2 - 4ac$. Then

1. if $d < 0$, the equation has no solutions;
2. if $d \geq 0$, the equation has the solutions $x = \dfrac{-b \pm \sqrt{d}}{2a}$.

Proof

If we combine theorems 4.5 and 4.7, we find that the second degree polynomial

$$f(x) = ax^2 + bx + c$$

can be written as[4]

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(-\frac{d}{4a}\right) = a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{d}{4a}.$$

[4] We obtain the expression by inserting the two formulas $x_0 = -\frac{b}{2a}$ and $y_0 = -\frac{d}{4a}$ into the formula $f(x) = a(x - x_0)^2 + y_0$.

This means that the equation $ax^2 + bx + c = 0$ may be written as

$$a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{d}{4a} = 0$$

If $d \geq 0$, this equation can be rewritten:[5]

[5] If $d < 0$, the equation cannot be solved. Thus the sign of the discriminant determines if the equation has any solutions.

$$\begin{aligned} a \cdot \left(x + \frac{b}{2a}\right)^2 &= \frac{d}{4a} && \Leftrightarrow \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{d}{4a^2} && \Leftrightarrow \\ \left(x + \frac{b}{2a}\right)^2 &= \left(\frac{\sqrt{d}}{2a}\right)^2 . \end{aligned}$$

This equation corresponds to[6]

[6] The $\pm$ on the right hand side stems from the fact that $(-h)^2 = h^2$, which means that equation $x^2 = h^2$ has both h and $-h$ as solutions.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{d}}{2a}.$$

Here we solve for x and get

$$\begin{aligned} x + \frac{b}{2a} &= \pm\frac{\sqrt{d}}{2a} && \Leftrightarrow \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{d}}{2a} && \Leftrightarrow \\ x &= \frac{-b \pm \sqrt{d}}{2a}. \end{aligned}$$

This proves the formula. ∎

If we look at the formula in theorem 4.9, we see that if $d = 0$, there is in fact only one solution since[7]

$$\frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}\,.$$

In this case, we say that the polynomial has a *double root*. This corresponds to the parabola touching the x-axis in exactly one point (see figure 4.6).

[7] If the discrminant is 0, the solution to the equation is $x = -\frac{b}{2a}$.

Here are a few examples of uses of the formula:

Example 4.10
To find the roots of the second degree polynomial

$$f(x) = 2x^2 + 2x - 12\,,$$

we find the coefficients of the polynomial:

$$a = 2, \quad b = 2 \quad \text{og} \quad c = -12\,.$$

Then we calculate the disciminant

$$d = b^2 - 4ac = 2^2 - 4\cdot 2\cdot(-12) = 100\,.$$

Since d is positive, there are two roots which are calculated in this manner:

$$x = \frac{-b \pm \sqrt{d}}{2a} = \frac{-2 \pm \sqrt{100}}{2\cdot 2} = \frac{-2 \pm 10}{4}\,.$$

So the two roots are

$$x = \frac{-2-10}{4} = -3 \quad \text{and} \quad x = \frac{-2+10}{4} = 2\,,$$

which we also see in figure 4.7.

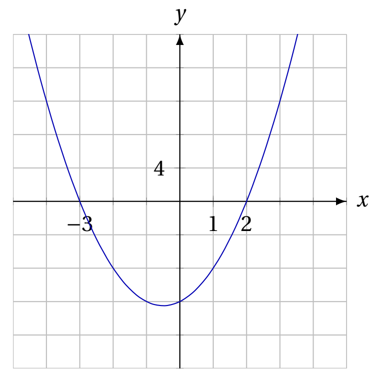


Figure 4.7: The polynomial $f(x) = 2x^2+2x-12$ has the roots -3 and 2.

Example 4.11
Here we solve the equation

$$-x^2 + 8x - 16 = 0\,.$$

The coefficients are

$$a = -1\,, \quad b = 8 \quad \text{and} \quad c = -16\,,$$

and the discriminant is

$$d = b^2 - 4ac = 8^2 - 4\cdot(-1)\cdot(-16) = 0\,.$$

Therefore the equation has the solution

$$x = -\frac{b}{2a} = -\frac{8}{2\cdot(-1)} = 4\,.$$

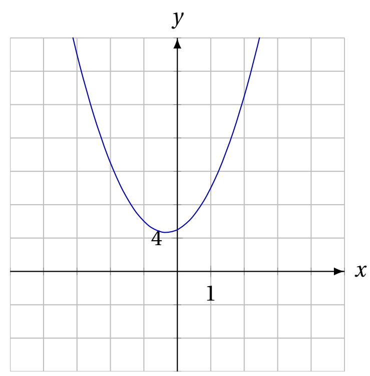


Figure 4.8: The polynomial $f(x) = 3x^2+2x+1$ has no roots.

Example 4.12
To find the roots of the second degree polynomial

$$f(x) = 3x^2 + 2x + 5\,,$$

we calculate the discriminant

$$d = b^2 - 4ac = 2^2 - 4\cdot 3\cdot 5 = 4 - 60 = -56\,.$$

Since the discriminant is negative, the polynomial has no roots. This is illustrated in figure 4.8.

Simple Quadratic Equations

As it turns out, it is not always necessary to use the quadratic formula to solve a quadratic equation. If either b or c equals 0, the equations can be solved in a much easier way.

Example 4.13 ($b = 0$)
In the quadratic equation

$$3x^2 - 75 = 0\,,$$

the coefficient $b = 0$. This equation can be solved in the following way:

$$\begin{aligned} 3x^2 - 75 &= 0 && \Leftrightarrow \\ 3x^2 &= 75 && \Leftrightarrow \\ x^2 &= 25 && \Leftrightarrow \\ x &= \pm\sqrt{25} && \Leftrightarrow \\ x &= \pm 5\,. \end{aligned}$$

Here we only need to remember that there are two possible solutions when taking the square root: A positive and a negative.

Example 4.14 ($c = 0$)
In the quadratic equation

$$2x^2 + 14x = 0\,,$$

$c = 0$. Here we can solve the equation by factoring out x and using the zero product property

$$\begin{aligned} 2x^2 + 14x &= 0 && \Leftrightarrow \\ 2x \cdot x + 7 \cdot 2x &= 0 && \Leftrightarrow \\ 2x \cdot (x + 7) &= 0 && \Leftrightarrow \\ 2x = 0 \quad \vee \quad x + 7 &= 0 && \Leftrightarrow \\ x = 0 \quad \vee \quad x &= -7\,. \end{aligned}$$

4.4 INTERPRETING THE COEFFICIENTS

There is a correlation between the graph of a second degree polynomial $f(x) = ax^2 + bx + c$ and the coefficients a, b, c, and the discriminant d. In this section, we describe these relations.

In section 4.2 above, we described how the sign of a determines whether the parabola opens upward or downward.[8]

[8] The argument only concerned the polynomial $p(x) = ax^2$, but since $f(x) = ax^2 + bx + c$ is $p(x)$ shifted, the argument also applies here.

The x-coordinate of the vertex is $x_0 = -\frac{b}{2a}$. If a and b have the same sign, x_0 must therefore be negative. In this case, the vertex is to the left of the y-axis. By the same argument, the vertex must be to the right of the y-axis if a and b have different signs. If $b = 0$, we get $x_0 = -\frac{0}{2a} = 0$; here the vertex is *on* the y-axis.

If we insert $x = 0$ into the formula for a general second degree polynomial, we get

$$f(0) = a \cdot 0^2 + b \cdot 0 + c\,,$$

i.e. the parabola passes through the point $(0, c)$. c must therefore be the y-intercept.

The correlation between the discriminant and the parabola was described in the previous section. Here we saw that when $d < 0$, the parabola does not intercept the x-axis, etc.

All in all, we have the following theorem:

Theorem 4.15

Let f be a second degree polynomial given by the formula

$$f(x) = ax^2 + bx + c\,,$$

and let d be the discriminant.

Then the graph of the polynomial is a parabola, and the following holds:

1. If $a > 0$ the parabola opens upward. If $a < 0$ it opens downward.
2. If a and b have the same sign, the vertex is to the left of the y-axis. If their signs are opposite, the vertex is to the right of the y-axis; and if $b = 0$, the vertex is on the y-axis.
3. The parabola intercepts the y-axis at $(0, c)$.
4. If $d > 0$, the parabola intercepts the x-axis twice. If $d < 0$ the parabola does not intercept the x-axis; and if $d = 0$, the parabola touches the x-axis in one point.

4.5 FACTORING

If a second degree polynomial $f(x) = ax^2 + bx + c$ has two roots r_1 og r_2, these are

$$r_1 = \frac{-b + \sqrt{d}}{2a} \qquad \wedge \qquad r_2 = \frac{-b - \sqrt{d}}{2a}\,.$$

It turns out that in this case, the polynomial can also be written as

$$f(x) = a(x - r_1)(x - r_2)\,.$$

We call this the *factored* polynomial.[9]

[9]That this is true can be demonstrated by reducing

$$a\left(x - \frac{-b + \sqrt{d}}{2a}\right)\left(x - \frac{-b - \sqrt{d}}{2a}\right)$$

to $ax^2 + bx + c$.

Example 4.16
If we want to factor the second degree polynomial $f(x) = 3x^2 - 3x - 6$, we first need to find the roots. To do this, we calculate the discriminant

$$d = b^2 - 4ac = (-3)^2 - 4 \cdot 3 \cdot (-6) = 81\,.$$

The two roots are therefore

$$r_1 = \frac{-b + \sqrt{d}}{2a} = \frac{-(-3) + \sqrt{81}}{2 \cdot 3} = 2$$

and

$$r_2 = \frac{-b - \sqrt{d}}{2a} = \frac{-(-3) - \sqrt{81}}{2 \cdot 3} = -1 \, .$$

The polynomial can now be factored:

$$\begin{aligned} f(x) &= a(x - r_1)(x - r_2) \\ &= 3(x-2)(x-(-1)) \\ &= 3(x-2)(x+1) \, . \end{aligned}$$

When a polynomial is factored, we can find the roots directly in the formula.[10] If a polynomial has no roots it cannot be factored.

The two second degree polynomials $f(x) = ax^2 + bx + c$ and $p(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ must have the same roots.[11] Factoring $p(x)$ must yield

$$p(x) = (x - r_1)(x - r_2) \, ,$$

since the coefficient of x^2 is 1.

If we multiply the parentheses we get

$$p(x) = x^2 - r_1 x - r_2 x + r_1 r_2 = x^2 - (r_1 + r_2)x + r_1 r_2 \, .$$

This polynomial must be the same as the original, i.e. the coefficients must be the same. Therefore

$$-\frac{b}{a} = r_1 + r_2 \qquad \wedge \qquad \frac{c}{a} = r_1 r_2 \, .$$

Since the polynomial $f(x) = ax^2 + bx + c$ has the same roots, these equations must also apply to this polynomial. We can use this to guess the roots of a second degree polynomial.

Example 4.17
If we want to try to guess the roots of the polynomial $f(x) = 4x^2 - 12x + 8$, we start by calculating

$$-\frac{b}{a} = -\frac{-12}{4} = 3 \, ,$$

and

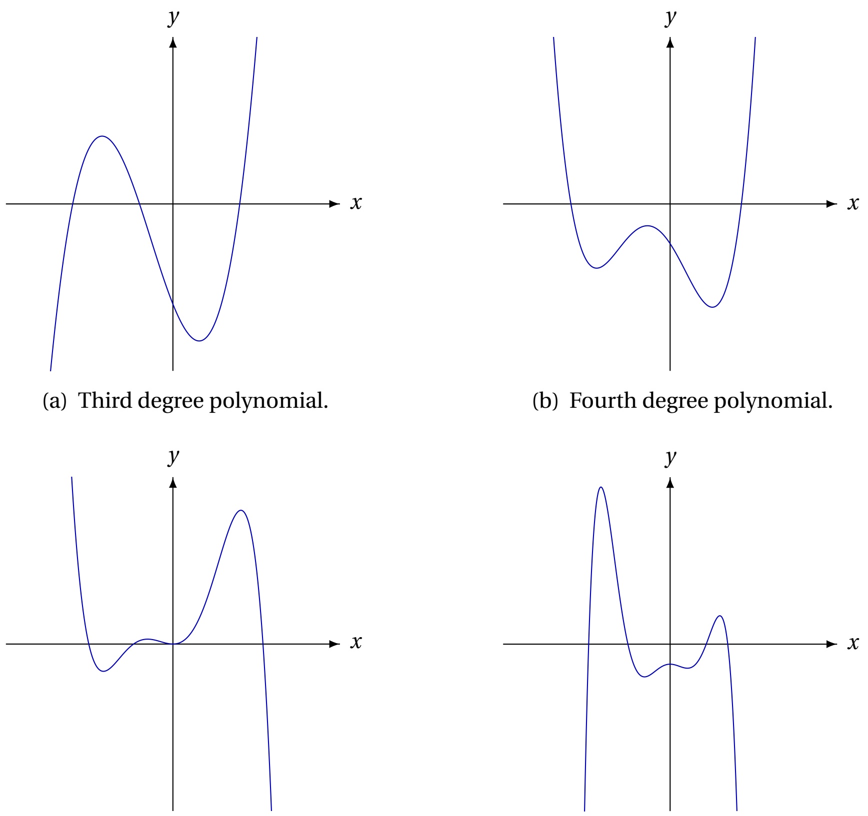
$$\frac{c}{a} = \frac{8}{4} = 2 \, .$$

From the previous arguments, we see that the sum of the roots $(r_1 + r_2)$ must be 3, while the product $(r_1 r_2)$ must be 2. This is only true for the numbers 1 and 2 and therefore these two numbers are the roots.

We sum up our results in the following theorem:

[10] If a second degree polynomial has only one root r, the factored polynomial will be $f(x) = a(x - r)^2$ where r is the double root.

[11] This is because if $ax^2 + bx + c = 0$ then we must also have $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.



(a) Third degree polynomial.

(b) Fourth degree polynomial.

(c) Fifth degree polynomial.

(d) Sixth degree polynomial.

Figure 4.9: The graphs of four polynomials with a degree larger than two. As can be seen from the figure, the graphs turn more often the higher the degree of the polynomial.

Theorem 4.18

If a second degree polynomial $f(x) = ax^2 + bx + c$ has at least one root, it can be factored, i.e.

$$f(x) = a(x - r_1)(x - r_2)\,,$$

where r_1 and r_2 are the two roots (if there is only one root then $r_1 = r_2$).

For the two roots, we also have

$$r_1 + r_2 = -\frac{b}{a}\,,$$

and

$$r_1 r_2 = \frac{c}{a}\,.$$

4.6 POLYNOMIALS OF A HIGHER DEGREE

Polynomials of a degree larger than 2 are not as easily described. In figure 4.9, we see the graph of polynomials of degree 3 to 6. As we see in the figure, the graphs turn more often, the higher the degree of the polynomial. The “turning points” are called the *extrema* of the polynomial.[12]

[12] An *extremum* is a collective term for a maximum or a minimum.

Since the graphs may have many extrema, it is also possible for the graph to intercept the x-axis many times. Therefore it is not possible to derive simple formulas for the roots.[13]

[13] There are formulas to calculate the roots of third and fourth degree polynomials; but they are complicated. For polynomials of degree five or more, it can be shown that no such formula can exist.[2]

In the previous section, it was described how we may factor a second degree polynomial. It is possible to factor polynomials of a higher degree as well—if we know the roots. In general, if r is a root in the polynomial $p(x)$ then $p(x)$ can be written as

$$p(x) = (x - r) \cdot q(x) \,,$$

where $q(x)$ is also a polynomial. The degree of q is 1 less than the degree of p. E.g. if p is a fourth degree polynomial then q is a third degree polynomial.

From this we infer that a polynomial of degree n may have at most n roots. And since solving an equation of degree n corresponds to finding a root of a polynomial of degree n, we have the following theorem:

Theorem 4.19

The following holds:

1. A polynomial of degree n has at most n roots.
2. An equation of degree n has at most n solutions.

Since there is no general solution formula to determine the roots of a polynomial of degree n, we must use other methods. CAS systems usually have a built in "factor"-function which can be used to factor polynomials.

Example 4.20
The polynomial $f(x) = 2x^3 - 16x^2 + 2x + 84$ can be written as (using a CAS)

$$f(x) = 2 \cdot (x+2) \cdot (x-3) \cdot (x-7) \,.$$

Here we see straight away that this third degree polynomial has the roots -2, 3 and 7.

Example 4.21
The fourth degree polynomial $f(x) = x^4 - x^3 - 19x^2 - x - 20$ can be factored, and we get

$$f(x) = (x-5) \cdot (x+4) \cdot (x^2+1) \,.$$

This polynomial has the two roots -4 and 5.

The polynomial cannot be factored further. This is because the second degree polynomial $x^2 + 1$ has no roots. For the same reason, the polynomial f has only two roots, despite the fact that it is a fourth degree polynomial.

In the example above, we saw a fourth degree polynomial with only two roots. It is actually possible to construct a fourth degree polynomial with no roots at all. However, this is not possible for a fifth degree polynomial. The reason for this might be glimpsed from figure 4.9. A polynomial of odd degree will always have at least one root.

It is also possible to use factoring to construct a polynomial with specific roots. This is shown in the concluding example:

Example 4.22
A third degree polynomial with the roots -1, 4 and 7 could be

$$f(x) = (x - (-1)) \cdot (x - 4) \cdot (x - 7) .$$

If we multiply and reduce, this polynomial may also be written as

$$f(x) = x^3 - 10x^2 + 17x + 28 .$$

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