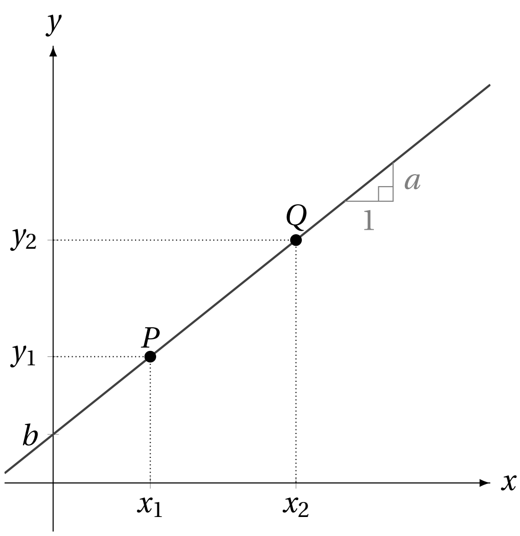
MATHEMATICS
BASIC COURSE

Mike Auerbach
www.mathematicus.dk



Mathematics: Basic Course
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Preface

This document is a translation of the Danish “Matematik: Grundforløbet”, which is a textbook on mathematics for the basic level of the Danish stx. Since English is not my first language, I apologise in advance for errors in translation.

The primary aim is to provide a textbook without too much “clutter”. Examples are kept to a minimum, and the text mainly covers the basic mathematics. It would therefore be a good idea to supplement the text with examples and other materials that cover specific uses of the mathematical tools.

Mike Auerbach

ORIGINAL PREFACE (IN DANISH)

Disse matematiknoter er skrevet til matematikundervisningen i grundforløbet på stx. Noterne er skrevet med det formål at have en grundbog, som kun indeholder den grundliggende matematiske teori. I forbindelse med samarbejde i studieretningen eller med andre fag er det derfor nødvendigt at supplere med eksempler og andet materiale, der dækker konkrete anvendelser.

Til gengæld dækker noterne den rent matematiske fremstilling af kernestoffet på stx, hvilket ifølge min opfattelse gør dem velegnede til en første behandling af stoffet samt i forbindelse med eksamenslæsningen.

Til slut en stor tak til de mange matematikkolleger, der er kommet med rettelser og gode ændringsforslag. De fejl og mangler, der stadig måtte findes, er naturligvis udelukkende mit ansvar.

Mike Auerbach
Tornbjerg Gymnasium

Contents

1 Graphs and Functions

1.1 VARIABLES

A *variable* is—as the name implies—a quantity, which varies.[1] It is a quantity that does not have a fixed value. A mathematical variable is always denoted by exactly one letter,[2] e.g. x.

We often encounter variables in formulas. A formula is an expression used to calculate a certain quantity based on other known quantities.

An example is the formula for the area of a rectangle. We have

$$A = l \cdot w \,, \tag{1.1}$$

where A is the area, l is the length, and w is the width of the rectangle. In this example, A, l and w are variables, i.e. they do not have a fixed value. But there is a *connection* between the values of the variables—which is expressed in the formula.

Mathematics is especially useful for exploring this sort of connections. We might look at the connection between areas and lengths, the connection between the price and demand, or the connection between travel time and the distance travelled.

Example 1.1
A connection to a domestic gas supply costs DKK 937.50 annually, plus DKK 7.26 per m^3 of gas used.

We can describe the cost of the gas connection in the formula

$$P = 7.26 \cdot V + 937.50 \,,$$

where P is the price paid, and V is the used amount of m^3 of gas.

Example 1.2
If we let an object drop, the connection between the time the object has fallen, and the distance is

$$s = 4.91 \cdot t^2 \,,$$

where t is the time (in seconds), and s is the distance fallen (in metres). [3]

[1] A quantity, which has a fixed value, is called a *constant*.

[2] In mathematics, lower and upper case letters denote different variables—i.e. x and X are *different* variables.

[3] This correlation was found experimentally by Galileo Galilei at the end of the 1500s.[5]

Formula (1.1) shows us that the area of a rectangle depends on its length and width. l and w are here called *independent* variables because their values may vary freely, while A is called the *dependent* variable, since its value depends on the values of the other two variables.

If we instead want to view l as a dependent variable, the formula (1.1) may be rewritten as[4]

$$l = \frac{A}{w} \,.$$

[4]In a mathematical context, we usually view a variable isolated on one side of the equation as the dependent variable, and the rest as independent.

1.2 PROPORTIONALITY

Variables may be connected in such a way that one is *proportional* to the other. We define the concept of *proportionality* in the following way:

Definition 1.3: Proportionality

If c is a constant ($c \neq 0$), then

1. y is *directly proportional* to x if $y = c \cdot x$,
2. y is *inversely proportional* to x if $y = \frac{c}{x}$.

The constant c is called the *proportionality constant*.

When we talk about direct proportionality, we often leave out "direct". I.e., if we write "y is proportional to x", we actually mean "... directly proportional to ...".

If we rewrite the formulas in defintion 1.3, the two forms of proportionality can also be expressed in the following way:

1. y is directly proportional to x if $\frac{y}{x} = c$.
2. y is inversely proportional to x if $y \cdot x = c$.

We also see that if $y = c \cdot x$, then $x = \frac{1}{c} \cdot y$. So if y is directly proportional to x (with proportionality constant c), x is also directly proportional to y (with proportionality constant $\frac{1}{c}$).

If $y = \frac{c}{x}$, then $x = \frac{k}{y}$. So if y is inversely proportional to x, then x is also inversely proportional to y (with the same proportionality constant).

Example 1.4
If a telephone call costs DKK 0.70 per minute, the price of a call is directly proportional to the duration.

If the price is called P, and the duration (in minutes) is called D, then

$$P = 0.70 \cdot D \,.$$

Here, the proportionality constant is 0.70.

Example 1.5
If we drive from Odense to Copenhagen (a distance of about 160 km), the travel time is inversely proprtional to the speed, with which we travel.

If we measure the time t in hours, and the speed v in kilometres per hour, the proportionality constant is 160, and

$$t = \frac{160}{v}\,.$$

From this we calculate that if we drive with a speed of 80 km/h, we get to Copenhagen from Odense in 2 hours, whereas if we drive with a speed of 160 km/h, we get there in only 1 hour.

Example 1.6
"T is proportional to the square of p and inversely proportional to s."

We can write this as the formula

$$T = c\cdot\frac{p^2}{s}\,,$$

where c is the proportionality constant.

1.3 GRAPHS

If we have two variables, where the value of one of the variables depends on the other, it is possible to draw a figure illustrating this dependency. We call this a *graph*,.

Coordinate Systems

A coordinate system is a sort of grid, which we lay across the plane. It is used to describe the position of points.

We start by drawing two perpendicular axes, the x- and the y-axis. The two axes are actually number lines, which intersect at 0, see figure 1.1. The intersection between the two axes is called the *origin* of the coordinate system.

If we draw a line from a point in the plane perpendicular to the x-axis, the line intersects the x-axis at a number. We call this number the *x-coordinate* of the point. Similarly, we define the *y-coordinate* as the number we get where a line perpendicular to the y-axis intersects the axis. Any point can now be described by its coordinates, i.e. a point is a pair of coordinates (x, y), which tell us, where the point is placed in the coordinate system (see figure 1.1). The origin has coordinates $(0,0)$.

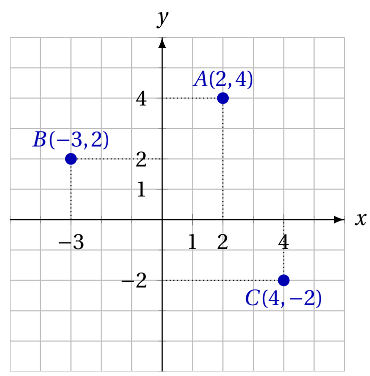

Figure 1.1: The three points $A(2,4)$, $B(-3,2)$ and $C(4,-2)$ in a coordinate system.

Graphs

If there is a dependency between two variables (e.g. x and y), we can draw a *graph* to illustrate this. We do this by marking each of the points (x, y), where the coordinates x and y fit the dependency.

Example calculation Here, we look at the dependency

$$y = 5 - x^2\,.$$

Table 1.1: A table of $y=5-x^2$.

x	y
-2	1
-1	4
0	5
1	4
2	1
3	-4

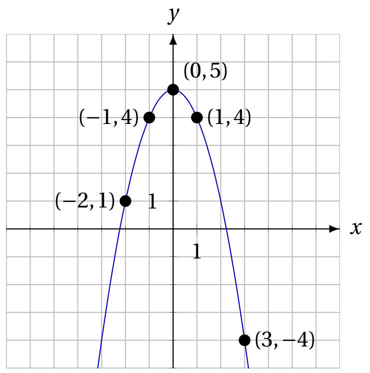

Figure 1.2: The graph of $y=5-x^2$.

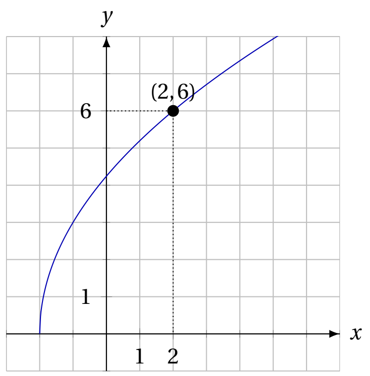

Figure 1.3: The graph of $f(x)=3\cdot\sqrt{x+2}$.

We now calculate a table of values of x and corresponding values of y. E.g. if we choose $x=3$, we get:

$$x=3 \qquad \Rightarrow \qquad y=5-3^2=-4\,.$$

The finished table could look like 1.1.

This table shows that some of the points in the graph are $(-2,1)$, $(-1,4)$, $(0,5)$, etc. We now draw these points in a coordinate system, and the graph is drawn by connecting the points with a curve (see figure 1.2).

Using the graph, we can find pairs of numbers x and y, which "go together". We can of course also do this by calculation, using the formula $y=5-x^2$.

1.4 FUNCTIONS

In a formula such as $y=5-x^2$, the right hand side is an expression, whose value only depends on the value of the variable x. Therefore we say that the right hand side is a *function of* x.

Functions are named using a letter, here we will call it f. To show that f is a function of x, we write:

$$f(x)=5-x^2\,.$$

This means that f represents the calculation "square the number, and subtract it from 5".

$f(x)$ is read "f of x", and the x in the parentheses tells us that the value of f (the *function value*) depends om the value of x.

We call $f(x)=5-x^2$ a formula for the function. The formula shows us how to calculate the function value for a given value of the independent variable.

Example calculation Here we look at the function

$$f(x)=3\cdot\sqrt{x+2}\,.$$

The graph of this function can be seen in figure 1.3.

As we see in the figure, the graph of the function passes through the point $(2,6)$. We write this in the following way:

$$f(2)=6\,.$$

This means that if we put $x=2$, the function value will be 6. We can also calculate this by replacing x in the formula by 2:

$$f(2)=3\cdot\sqrt{2+2}=3\cdot\sqrt{4}=3\cdot 2=6\,.$$

We can also read the function value from the graph like in figure 1.3.

Example 1.7
Here, we look at the function $h(t)=t^2-3$.

The function values $h(-1)$ and $h(4)$ are calculated like this:

$$h(-1) = (-1)^2 - 3 = 1 - 3 = -2$$
$$h(4) = 4^2 - 3 = 16 - 3 = 13\,.$$

This means that the graph of the function h passes through the two points $(-1, -2)$ and $(4, 13)$.

If we know the function value, we can also go backwards and find out, which value of the independent variable has this function value. We can use the graph for this, but the problem can also be solved by calculation, like the following example shows.

Example 1.8
When does the function $g(x) = 2x + 1$ assume the value 17?

The answer to this question may be found by solving the equation $g(x) = 17$. This is done in the following way:

$$\begin{aligned} g(x) &= 17 &&\Leftrightarrow \\ 2x + 1 &= 17 &&\Leftrightarrow \\ 2x &= 16 &&\Leftrightarrow \\ x &= 8\,. \end{aligned}$$

The answer to the question is that $g(x) = 17$ when $x = 8$.

1.5 INTERSECTIONS

If two graphs intersect, the intersections can be found by equating the two functions' formulas and solving the equation.

This works because at the points, where the graphs intersect, the function value and the value of the dependent variable must be equal for both functions. We illustrate this by an example.

Example 1.9
The two functions

$$f(x) = x - 5 \qquad \text{and} \qquad g(x) = -2x + 1$$

have intersecting graphs (see figure 1.4).

The coordinates of the intersection can be found by solving the equation $f(x) = g(x)$:

$$\begin{aligned} x - 5 &= -2x + 1 &&\Leftrightarrow \\ 3x &= 6 &&\Leftrightarrow \\ x &= 2\,. \end{aligned}$$

Now we know the x-coordinate. To find the intersection, we also need to know the y-coordinate. This is determined by inserting the x-coordinate in one of the functions:

$$y = f(5) = 2 - 5 = -3\,.$$

The two graphs intersect at $(2, -3)$, which we can also see in figure 1.4.

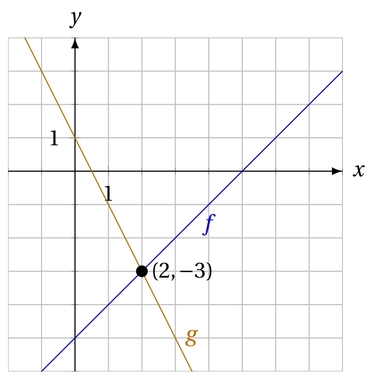

Figure 1.4: The intersection of the graphs of the two functions $f(x) = x - 5$ and $g(x) = -2x + 1$.

1.6 SOLVING EQUATIONS GRAPHICALLY

We can find intersection points between graphs by solving an equation. This means that we can also solve equations by finding intersection points.

If we have an equation, the two sides may be viewed as functions. Where the two sides are equal (i.e. where the graphs intersect), we find the solution(s) of the equation.

Example 1.10
The equation

$$x^2 - 3 = -2x$$

can be solved by drawing the graphs of

$$f(x) = x^2 - 3 \qquad \text{and} \qquad g(x) = -2x\,.$$

In figure 1.5, we see that the two graphs intersect at $(-3, 6)$ and $(1, -2)$. This equation therefore has two solutions:

$$x = -3 \qquad \vee \qquad x = 1\,.$$

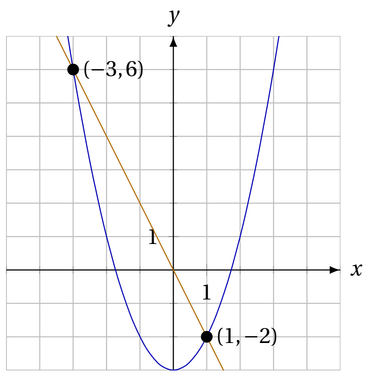

Figure 1.5: The solutions of $x^2 - 3 = -2x$ are the x-coordinates of the intersections.

Linear Functions 2

Functions can be described by formulas. If functions have formulas that follow a similar pattern, they have similar properties.[1] A similar pattern might be

$$f(x) = 3x + 2, \qquad g(x) = 7x - 5 \qquad \text{and} \qquad h(x) = -4x + 3\,.$$

[1] This also applies if they have similar graphs.

Functions whose formulas follow a pattern like f, g and h above are called *linear functions*.

Definition 2.1

A *linear function* is a function of the form

$$f(x) = ax + b\,,$$

where a and b are two numbers.

It turns out that the graph of a linear function is a straight line. This is one of the reasons why these functions are called linear (see figure 2.1).

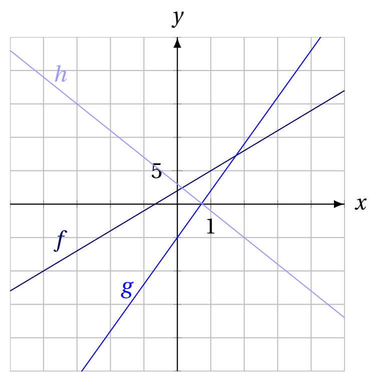

Figure 2.1: The graphs of the three linear functions f, g and h.

2.1 SLOPE AND INTERCEPT WITH THE AXES

From figure 2.1, we can find the values of the numbers a and b in the formulas. We have the following theorem:

Theorem 2.2

For a linear function $f(x) = ax + b$ the following holds:

1. If the independent variable x increases by 1, the function value $f(x)$ increases by a.
2. The graph of the function intercepts the y-axis at b.

Proof
When x increases by 1, the function value increases from $f(x)$ to $f(x+1)$. Therefore the function value increases by

$$f(x+1) - f(x) = (a(x+1) + b) - (ax + b)$$

$$= ax + a + b - ax - b$$
$$= a\,.$$

On the y-axis $x = 0$. The intercept with the y-axis is therefore

$$f(0) = a \cdot 0 + b = b\,. \qquad \blacksquare$$

This theorem shows that for linear functions, the function value increases by a fixed number (a) every time x increases by 1. This is the reason why the graph is a straight line. The larger the number a, the more $f(x)$ increases, and the graph is steeper. Therefore the number a is called the *slope* of the graph..

If a is a negative number, $f(x)$ decreases when x increases; the function is then decreasing.

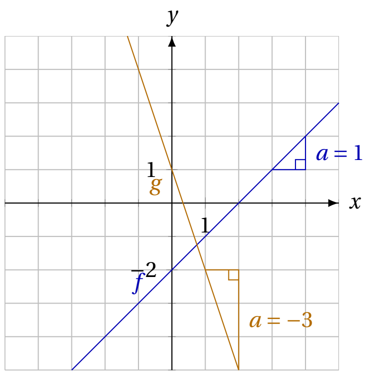

Figure 2.2: Finding the numbers a and b.

Theorem 2.3

The slope a of the function $f(x) = ax + b$ has the following property:

1. If $a > 0$ the function is increasing.
2. If $a < 0$ the function is decreasing.

Example 2.4
In figure 2.2 we see the graphs of the linear functions f and g.

The graph of f intercepts the y-axis at -2. If we move 1 to the right of the graph, we have to move 1 up to reach the graph again. Therefore $a = 1$.

Thus f has the formula $f(x) = 1 \cdot x + (-2)$ or

$$f(x) = x - 2\,.$$

The graph of g intercepts the y-axis at 1, and if we move 1 to the right of the graph, we need to move 3 down; so $a = -3$. The formula is

$$g(x) = -3x + 1\,.$$

In the special case where the slope is 0, the function is constant, i.e. the graph is a line parallel to the x-axis. Such a line does not intercept the x-axis.[2]

[2] Unless the line *is* the x-axis, in which case it intercepts the x-axis in an infinite number of points.

However, a linear function whose slope is not 0 does have a graph that intercepts the x-axis. This intercept can be calculated from the formula.

Theorem 2.5

The graph of the linear function $f(x) = ax + b$ intercepts the x-axis in the point $\left(-\frac{b}{a}, 0\right)$.

Proof
On the x-axis the y-coordinate is 0.[3] The graph of f must therefore inter-

[3] Remember that every point on the x-axis can be written as $(x, 0)$ and every point on the y-axis can be written as $(0, y)$.

cept the x-axis where

$$f(x) = 0\,,$$

i.e.

$$ax + b = 0 \quad \Leftrightarrow \quad ax = -b \quad \Leftrightarrow \quad x = -\frac{b}{a}\,.$$

Thus the intercept with the x-axis is $\left(-\frac{b}{a}, 0\right)$. ■

Example 2.6

The linear function $f(x) = 4x - 12$ intercepts the y-axis in $b = -12$ and has slope $a = 4$. It intercepts the x-axis in

$$x = -\frac{b}{a} = -\frac{-12}{4} = 3\,.$$

Therefore the intercept with the x-axis is the point $(3, 0)$, and the intercept with the y-axis is the point $(0, -12)$.

2.2 CALCULATING THE FORMULA

If we know two points on the graph of a linear function, this suffices to uniquely determine the function.[4] There is a correlation between the coordinates of the points and the numbers a and b.

[4] This is because precisely one straight line passes through two given points.

It turns out that this correlation can be described by two simple formulas.

Theorem 2.7

If the graph of $f(x) = ax + b$ passes through the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ then

$$a = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{and} \qquad b = y_1 - ax_1\,.$$

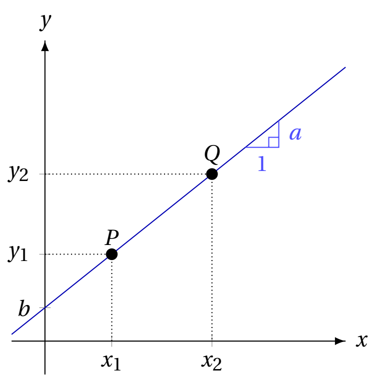

Figure 2.3: The two points P and Q on the graph of $f(x) = ax + b$.

Proof

In figure 2.3, we see the graph of the function $f(x) = ax + b$ and the two points $P(x_1; y_1)$ and $Q(x_2; y_2)$.

Because the point P is on the line $f(x_1) = y_1$, and since Q is on the line $f(x_2) = y_2$. This yields the equations

$$\begin{aligned} y_2 &= ax_2 + b\,, \\ y_1 &= ax_1 + b\,. \end{aligned} \tag{2.1}$$

If we subtract the second equation from the first we get

$$y_2 - y_1 = (ax_2 + b) - (ax_1 + b)\,,$$

which can be reduced to

$$y_2 - y_1 = ax_2 - ax_1\,.$$

On the right hand side, a is a common factor so

$$y_2 - y_1 = a(x_2 - x_1) \quad \Leftrightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = a\,.$$

This proves the formula for a.

To obtain the formula for b, we again look at equation (2.1):

$$y_1 = ax_1 + b\,.$$

If we solve for b in this equation we get

$$y_1 - ax_1 = b\,,$$

which proves the formula for b. ■

Example 2.8
The linear function f has a graph that passes through the points $P(3,5)$ and $Q(6,-7)$. What is the formula for the function?

To answer this question, we look at the two points. Here we have

$$x_1 = 3\,,\quad y_1 = 5\,,\quad x_2 = 6 \quad \text{and} \quad y_2 = -7\,.$$

Now we use the formulas in theorem 2.7 and get

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-5}{6-3} = \frac{-12}{3} = -4\,,$$

and

$$b = y_1 - ax_1 = 5 - (-4)\cdot 3 = 5 + 12 = 17\,.$$

The function therefore has the formula $f(x) = -4x + 17$.

2.3 LINEAR GROWTH

The growth of a linear function can be described in the following way:[5]

[5]This theorem is an extension of theorem 2.2.

Theorem 2.9

Let f be a linear function, $f(x) = ax + b$. When x increases by Δx the function value increases by $a\cdot\Delta x$.

Proof
If x increases from x_1 to x_2 where $x_2 = x_1 + \Delta x$ then the function value will increase from

$$y_1 = f(x_1) = ax_1 + b$$

to

$$y_2 = f(x_2) = f(x_1 + \Delta x) = a(x_1 + \Delta x) + b = ax_1 + a\cdot\Delta x + b\,.$$

The function value then increases by

$$y_2 - y_1 = (ax_1 + a\cdot\Delta x + b) - (ax_1 + b) = a\cdot\Delta x\,.$$

This proves the theorem. ■

Example 2.10
In table 2.1, we see an example of the growth of a linear function.

The function $f(x) = 3x + 7$ is increasing, and every time x increases by 2, the function value increases by $3\cdot 2$.

Example 2.11
Here we look at the function $f(x) = 3x - 4$ which has slope $a = 3$. If x increases by $\Delta x = 5$, the function value will increase by

$$a \cdot \Delta x = 3 \cdot 5 = 15\,.$$

Every time x increases by 5, the function value increases by 15.

We can also ask how much x must increase for the function value to increase by 60? In this case $a \cdot \Delta x = 60$, i.e.

$$3 \cdot \Delta x = 60 \qquad \Leftrightarrow \qquad \Delta x = 20\,.$$

x has to increase by 20, for the function value to increase by 60.

Table 2.1: The growth of $f(x) = 3x + 7$.

	x	y	
	-2	1	
$+2$			$+3\cdot 2$
	0	7	
$+2$			$+3\cdot 2$
	2	13	
$+2$			$+3\cdot 2$
	4	19	

Example 2.12
Here we look at the function $f(x) = -2x + 7$. When x increases by $\Delta x = 3$ the function value increases by

$$a \cdot \Delta x = -2 \cdot 3 = -6\,.$$

Since the function value increases by -6, it decreases by 6 every time x increases by 3.[6]

[6] A negative increase corresponds to a decrease. In mathematical models it is often useful to calculate a signed increase and determine afterwards if it is actually an increase or a decrease.

The next examples demonstrate how a mathematical description of linear growth can answer certain questions.

Example 2.13
In a certain town the population is given by the function

$$N(x) = 213x + 14\,752\,,$$

where $N(x)$ is the number of inhabitants x years after the year 2000.

In this formula there are two constants 213 and 14 752. The function $N(x)$ is a linear function, i.e. the number 213 is a slope: Every time x increases by 1 the function value increases by 213. Since x is measured in years and the function value is the population, we know that the number of inhabitants increase by 213 every time x increases by 1 year. Therefore the population grows by 213 inhabitants per year.

14 752 is the y-intercept. This is where $x = 0$ and this happens in the year 2000.[7] From this we deduce that the population of the city was 14 752 inhabitants in the year 2000.

[7] Since the year 2000 lies 0 years after the year 2000.

Example 2.14
Here we look at the same model as in example 2.13,

$$N(x) = 213x + 14.752\,.$$

How much does the population grow in a 10 year period?

From the formula we see that the population grows by 213 per year. In a 10 year period the increase in population will therefore be

$$10 \cdot 213 = 2130$$

inhabitants.

Example 2.15
A company produces a certain amount of products. The cost of production is a fixed cost of DKK 2000 and a cost per item of DKK 17.

This means the total cost of production is a function of the number of items produced. The formula is

$$c(x) = 17x + 2000\,,$$

where x is the number of items produced and $c(x)$ is the total cost of production.

Example 2.16
The average temperature in West Greenland depends on the latitude,[3]

$$T(x) = -0.732x + 46.1\,,$$

where T is the average temperature (in °C) and x is the latitude.

This means that the average temperature in West Greenland decreases by $0,732$°C per degree of latitude.

A quick interpretation of the number 46.1 would be that it is the temperature at latitude 0 degrees, i.e. at the Equator. However, the model only applies to West Greenland, so this interpretation does not make sense.

It is therefore not possible to give a meaningful interpretation of the number 46.1.

2.4 LINEAR REGRESSION

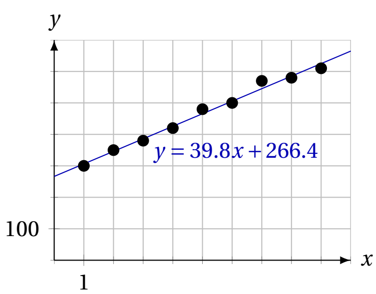

Figure 2.4: A series of data points and the straight line that is a best fit.

When we have a series of data points (measurements), we sometimes have a situation like the one in figure 2.4 where the data points do not lie exactly on a straight line—but do so approximately.

Since the points do not lie exactly on the line, it would be wrong to use theorem 2.7 to calculate a formula. Depending on the choice of points to use, we would get very different results for the formula.

Instead, we use a method called *linear regression* to determine which straight line is "closest" to all of the points. This method is built in to most spreadsheets and mathematical computer programs. We input the data points, and the program calculates the equation. The equation in figure 2.4 is found in this way.

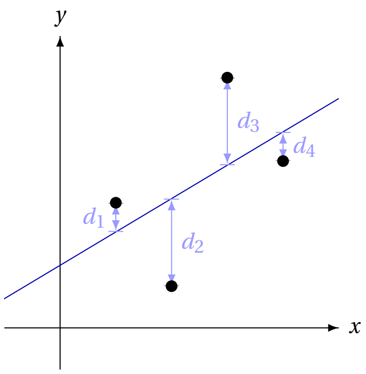

Figure 2.5: We minimise the square sum $D = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

The method involves finding the line which is closest to all of the points. The distance is defined as the *square sum* D of the vertical distance from the line to the points. In figure 2.5 this distance is

$$D = d_1^2 + d_2^2 + d_3^2 + d_4^2\,.$$

This sum contains more terms, the more data points we have. The best fit for a line is the one which minimises D. It is possible to derive formulas to calculate the equation for this line, but it is tedious work best left to a computer.

Percent and Interest 3

3.1 PRELIMINARY CONCEPTS

The word *percent* comes from the latin *centum*, which means “hundred”, so percent means “per hundred”. So e.g. 3% means “3 per hundred”, i.e. 3 hundredths or, to put it another way,

$$3\% = \frac{3}{100} = 0.03\,.$$

In calculations, we can therefore always replace the symbol % by a division by 100.[1]

[1] We should always do this because it removes a lot of confusion concerning the meaning of % in the calculation.

If we want to write a certain number as a percentage, we perform the opposite calculation. E.g.

$$0.72 = 0.72 \cdot \frac{100}{100} = \frac{0.72 \cdot 100}{100} = \frac{72}{100} = 72\%\,.$$

In this calculation, we multiply the number 0.72 by $\frac{100}{100}$, which is equal to 1.[2] The calculation looks a bit elaborate; if we notice that $\frac{100}{100}$ is actually 100%, we can instead write

[2] A number does not change when it is multiplied by 1—not even if the 1 is written as $\frac{100}{100}$.

$$0.72 = 0.72 \cdot 100\% = 72\%\,.$$

Here, we multiply 0.72 by 100%, which is just another way of multiplying by 1.

We often use the idea of percent to talk about fractions of a given quantity. The following theorem shows how to calculate percentages:

Theorem 3.1

p% of a given quantity A is

$$p\% \cdot A = \frac{p}{100} \cdot A\,.$$

Example 3.2
How much do you save if the price of a £80 jacket is lowered by 30%?

You will save 30% of £80, i.e.

$$30\% \cdot £80 = \frac{30}{100} \cdot £80 = 0.3 \cdot £80 = £24.$$

We may also want to know, how large a quantity is in relation to another. This is done in the following way:

Theorem 3.3

To find out, how many percent the quantity A_1 is in relation to the quantity A_0, we calculate

$$\frac{A_1}{A_0} \cdot 100\% \,.$$

Example 3.4
How large a percentage is 23 people out of 362?

To find out, we calculate

$$\frac{23}{362} = 0.0635 = 0.0635 \cdot 100\% = 6.35\% \,.$$

Example 3.5
How many percent is 465 out of 276?

Answer:
$$\frac{465}{276} = 1.6848 = 1.6848 \cdot 100\% = 168.48\% \,.$$

Here, we get a result above 100%, but this makes perfect sense since 465 is more than 276, so it has to be more than 100%.[3]

[3]It is important to remember that it is not the size of the numbers that determines, which number get divided by which; we always divide by the number with which we compare.

3.2 GROWTH IN PERCENT

In the preceding section, we looked at how to find a percentage of a given quantity, and how to compare two quantities. Here, we look at growth. How do we calculate the result if some quantity grows by a certain percent? And how do we find out, how much larger (or smaller) some quantity is in relation to another?

Calculated example If we want to add 12% to 140, we can do it in the following way:

1. First, we find 12% of 140:

$$12\% \cdot 140 = 0.12 \cdot 140 = 16.8 \,.$$

2. Then we add this to the original 140 and get:

$$140 + 16.8 = 156.8 \,.$$

This is actually a very elaborate way of doing it, especially if we want to add a certain percentage severeal times—e.g. if we want to find out, how much money is in a bank account after 1, 2, 3 or more years.

It turns out that the calculation above can be simplified quite a bit. If we combine the two steps, we see that in order to add 12% to 140, we need to calculate:

$$140 + 12\% \cdot 140 = 140 + 0.12 \cdot 140 \,.$$

If we factor out 140, we get

$$140 + 0.12 \cdot 140 = 140 \cdot (1 + 0.12) \,.$$

Here we se that in order to add 12% to 140, we need to multiply 140 by $1 + 0.12 = 1.12$.

We therefore have the following definition:

Definition 3.6

1. The *growth rate* r is the fraction a certain quantity grows.
2. The *growth factor* a is defined by

$$a = 1 + r.$$

Here are two examples of how to calculate the growth rate and the growth factor.

Example 3.7
If some quantity grows by 7.5% the growth rate is

$$r = 7.5\% = 0.075 \,,$$

and the growth factor is

$$a = 1 + r = 1 + 0.075 = 1.075 \,.$$

Example 3.8
If some quantity *decreases* by 11%, the growth rate is

$$r = -11\% = -0,11 \,,$$

and the growth factor is

$$a = 1 + (-0,11) = 0,89 \,.$$

Here, it is important to notice that when a quantity decreases, the growth rate is negative.

If we want to add 12% to 140 like we did before, we calculate

$$140 + 12\% \cdot 140 = 140 \cdot 1.12 \,.$$

This means that we actually just multiply 140 with the growth factor—which in this case is $a = 1.12$.

From this we get

Theorem 3.9

If we want to add a percentage to a quantity A_0, we get the new value

$$A_1 = A_0 \cdot a \,,$$

where $a = 1 + r$ is the growth factor.

In calculations concerning growth in percent—or when we want to know how much bigger/smaller a certain quantity is in relation to another—we never use the growth rate r directly, but instead use the growth factor a.

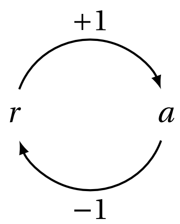

Figure 3.1: Conversion between growth rate and growth factor.

In any given context, we usually know only the growth rate r. We therefore need to calculate the growth factor a before proceeding with the calculations. If we instead are looking for a growth rate, we calculate the growth factor, and then convert this to a growth rate (see figure 3.1).

Here is an example where we add a percentage to a given quantity, using the theorem above:

Example 3.10
We want to add 25% VAT to an item, which costs DKK 399.96. What is the final price of this item?

Here, the growth rate is $r = 25\% = 0.25$. The growth factor is then

$$a = 1 + r = 1 + 0.25 = 1.25\,.$$

The final price is therefore

$$399.96 \cdot 1.25 = 499.95\,.$$

So the item costs DKK 499.95.

Now we look at how to compare two quantities:

Example 3.11
The price of an item drops from DKK 179.95 to 139.95. How many percent has the price decreased?

We can rewrite the formula $A_1 = A_0 \cdot a$ as $a = \frac{A_1}{A_0}$. Using this we calculate

$$a = \frac{A_1}{A_0} = \frac{139.95}{179.95} = 0.7777\,.$$

This is the growth factor, which means that the growth rate is

$$r = a - 1 = 0.7777 - 1 = -0.2223 = -22.23\%\,.$$

The price has therefore dropped by 22.23%.[4]

[4] In this example, we see clearly that we calculate using the growth factor, but present the growth rate in the conclusion.

Compound Interest

In the preceeding section, we explained how to add a certain percentage to a given quantity. Sometimes we need to add a percentage several times. E.g. if we deposit money into a bank account, how much money do we have in 2, 3 or more years?

Calculated example If we deposit DKK 10 000 into a bank account at an interest rate of 2.5% p.a.,[5] how much money do we have after 5 years?

[5] *per annum*, i.e. per year

According to theorem 3.9, we need to multiply the DKK 10 000 by the growth factor $a = 1 + 2.5\% = 1.025$ to find the amount of money after 1 year. This amount we multiply by 1.025 to find the amount after 2 years, etc. All in all, we need to multiply 10 000 by 1.025 five times:

$$10\,000 \cdot 1.025 \cdot 1.025 \cdot 1.025 \cdot 1.025 \cdot 1.025\,.$$

This means that we need to calculate

$$10\,000 \cdot 1.025^5 = 11\,314.08\,,$$

so after 5 years, we have DKK 11 314.08 in our account.

Adding interest in this way is called compounding. The formula for *compound interest* is given in the following theorem.

Theorem 3.12: Compound interest

A quantity P (called the *principal*), which increases by a growth rate of r per *compounding period*, increases to A_n after n periods, where

$$A_n = P \cdot a^n\,,$$

and $a = 1 + r$ is the growth factor. This formula can also be written as

$$A_n = P \cdot (1+r)^n\,.$$

We can answer several types of questions using this formula. A few are shown in the following examples.

Example 3.13

If we deposit \$ 10 000 into a bank account, what does the interest rate need to be p.a. for this amount to increase to \$ 12 000 in 10 years?

Here we know the principal $P = 10\,000$, the amount $A_{10} = 12\,000$, and the number of periods (years) $n = 10$. If we insert these values into the formula for compound interest, we get

$$A_n = P \cdot a^n \qquad \Rightarrow \qquad 12\,000 = 10\,000 \cdot a^{10}\,.$$

Now we solve the equation, and get

$$12\,000 = 10\,000 \cdot a^{10} \quad \Leftrightarrow \quad \frac{12\,000}{10\,000} = a^{10} \quad \Leftrightarrow \quad \sqrt[10]{\frac{12\,000}{10\,000}} = a.$$

a is then

$$a = \sqrt[10]{\frac{12\,000}{10\,000}} = 1.0184\,.$$

This is a growth factor, and we need the corresponding growth rate

$$r = 1.0184 - 1 = 0.0184 = 1.84\%\,.$$

If we want \$ 10 000 to increase to \$ 12 000 in 10 years, the interest rate needs to be 1.84% p.a.

Example 3.14
In a certain city, the growth rate has been 3% per year for the last 20 years. If there are 27 541 inhabitants in the city now, how many were there 12 years ago?

To solve this problem, we use the formula for compound interest and set $n = -12$, since we are going *back* 12 years.

We have $P = 27541$, $r = 3\%$, i.e. $a = 1.03$, and $n = -12$. The formula then gives us the answer:

$$K_{-12} = 27541 \cdot 1,03^{-12} = 19317\,.$$

12 years ago, the city had 19 317 inhabitants.

Example 3.15
Denmark has 5.6 million inhabitants. The growth rate is currently about 0.4% per year.[2] If this rate stays constant, how many years will pass before there are 6 million Danes?

In millions, $K_0 = 5,6$. The growth factor is $a = 1 + 0.4\% = 1.004$, and $K_n = 6$. The number of years, n, is unknown. We insert the known quantities into the formula for compound interest and get

$$K_n = K_0 \cdot a^n \qquad \Rightarrow \qquad 6 = 5.6 \cdot 1.004^n.$$

We solve the equation:[6]

[6]Remember that the equation $a^x = b$ has the solution $x = \frac{\log(b)}{\log(a)}$.

$$\begin{aligned} 6 &= 5.6 \cdot 1.004^n && \Leftrightarrow \\ \frac{6}{5.6} &= 1.004^n && \Leftrightarrow \\ \frac{\log\left(\frac{6}{5.6}\right)}{\log(1.004)} &= n && \Leftrightarrow \\ 17.3 &= n\,. \end{aligned}$$

In 17.3 years, the population will be 6 million if the growth rate stays at 0.4%.

3.3 CHANGING COMPOUNDING PERIODS

In this section, we will see how to convert between growth factors (and, by extension, growth rates) for different compounding periods, e.g. years and months.

Calculated example If a given quantity increases by 0.5% monthly, how do find the corresponding percentage increase per year?

The formula for compound interest tells us that in order to add interest to P for *one* month, we need to multiply by the growth factor

$$a_{\text{month}} = 1 + 0.5\% = 1.005\,.$$

Now, if we go forward one year, this corresponds to 12 months. According to the formula, we then need to multiply P by

$$a_{\text{month}}^{12} = 1.005^{12}\,.$$

The growth factor for one year is therefore

$$a_{\text{year}} = 1,005^{12} = 1,0617\,.$$

This corresponds to the growth rate $r_{\text{year}} = 1.0617 - 1 = 6.17\%$. A monthly rate of 0.5% thus corresponds to an annual rate of 6.17%.

Generalising this calculation gives us the following theorem:

Theorem 3.16

If the annual growth factor is a_{year}, and the monthly growth factor is a_{month}, then

$$a_{\text{year}} = a_{\text{month}}^{12}\,.$$

This theorem may be extended to other conversions than the one between months and years. The following examples illustrate this point.

Example 3.17
If some quantity P grows by 4% per year, how many percent does it grow in 10 years?

The annual growth factor is $a_{1\text{ year}} = 1 + 4\% = 1.04$. We can then calculate the growth factor for 10 years:

$$a_{10\text{ years}} = a_{1\text{ year}}^{10} = 1.04^{10} = 1.4802\,.$$

This corresponds to a growth rate of

$$r_{10\text{ years}} = 1.4802 - 1 = 0.4802 = 48.02\%\,.$$

If a quantity grows by 4% per year, it will grow by 48.02% in 10 years.

It is also possible to use theorem 3.16 to convert from years to months—i.e. in the opposite direction:

Example 3.18
If a bank pays 2.1% interest p.a., then what is the monthly interest rate?

First, we notice that since there are 12 months in a year, 1 month must correspond to $\frac{1}{12}$ of a year. Then we use theorem 3.16 like this:

$$a_{\text{month}} = a_{\text{year}}^{\frac{1}{12}}\,.$$

If we insert the annual growth rate $a_{\text{year}} = 1 + 2.1\% = 1.021$, we get

$$a_{\text{month}} = 1.021^{\frac{1}{12}} = 1.00173\,.$$

An annual interest rate of 2.1% therefore corresponds to a monthly interest rate of 0.173%.

3.4 AVERAGE GROWTH RATE

In every preceding section, we looked at a fixed growth rate. If the growth rate changes, what is it then possible to say about the growth?

Calculated example We deposit \$ 1000 into an account. The money stays in the account for 3 years, during which time the interest rate changes as shown in table 3.1.

Table 3.1: The interest r_i and the growth factor a_i for an account over 3 years.

Year	r_i	a_i
1	2.7%	1.027
2	3.0%	1.030
3	1.5 %	1.015

To determine how much money is in the account after 3 years, we need to know the corresponding growth factors. These are shown in the last column of table 3.1.

Now we can calculate the amount:

$$\begin{aligned} K_3 &= K_0 \cdot a_1 \cdot a_2 \cdot a_3 \\ &= 1000 \cdot 1.027 \cdot 1.030 \cdot 1.015 = 1073.68\,. \end{aligned} \tag{3.1}$$

So after 3 years, we have \$ 1073.68 in our account.

If the growth rate was fixed, what would it have to be in order for the account to have the same balance after 3 years? This fixed growth rate is an *average growth rate*.

To answer this question, we need to look at the formula for compound interest. If we denote the *average growth factor* by $\overline{a}$, then

$$K_3 = K_0 \cdot \overline{a}^3\,.$$

Since this calculation must have the same result—$K_3 = 1073.68$—as (3.1), we must have

$$\overline{a}^3 = a_1 \cdot a_2 \cdot a_3\,.$$

From this we get

$$\overline{a} = \sqrt[3]{a_1 \cdot a_2 \cdot a_3}\,.$$

In this case, the average growth factor becomes

$$\overline{a} = \sqrt[3]{1.027 \cdot 1.030 \cdot 1.015} = 1.02398\,.$$

The corresponding average growth rate is

$$\overline{r} = \overline{a} - 1 = 1.02398 - 1 = 2.398\%\,.$$

In general, we have the following theorem:

Theorem 3.19: Average growth rate

If a quantity increases through n periods by the different growth factors $a_1, a_2, \ldots, a_n$, the average growth factor per period is

$$\overline{a} = \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n}\,.$$

The average growth rate is therefore

$$\overline{r} = \sqrt[n]{(1+r_1) \cdot (1+r_2) \cdot \cdots \cdot (1+r_n)} - 1\,.$$

Example 3.20

The population of a town grows by changing rates in a 5-year period. The growth rates and factors can be seen in table 3.2.

Table 3.2: Growth rate r_i and growth factor a_i for the population in a town in year i.

i	r_i	a_i
1	2,2%	1,022
2	3,1%	1,031
3	−1,2%	0,988
4	4,2%	1,042
5	6,0%	1,060

r_3 is negative. This means that the population decreases in that year.

To find the average growth rate during the 5 years, we calculate the 5 growth factors (see table 3.2).

Now we can calculate the average growth factor:

$$\overline{a} = \sqrt[5]{1.022 \cdot 1.031 \cdot 0.988 \cdot 1.042 \cdot 1.060} = 1.02832\,.$$

The average growth rate during the 5 years is then

$$\overline{r} = 1.02832 - 1 = 0.02832 = 2.832\%\,.$$

Exponential Functions 4

Definition 4.1

An *exponential function* is a function of the form

$$f(x) = b \cdot a^x ,$$

where a and b are positive numbers.

The number a in definition 4.1 is called the *growth factor*[1] and b is the *initial value*.

[1] This is exactly the same factor as in the formula for compound interest $A_n = P \cdot a^n$.

We call b the initial value because the graph of the function intercepts the y-axis at $(0, b)$. This follows from the calculation

$$f(0) = b \cdot a^0 = b \cdot 1 = b .$$

A few graphs of exponential functions may be seen in figure 4.1. Graphs of exponential functions do not intercept the x-axis, because the function values are always positive, since a^x is always positive if a is positive—no matter which value x has.

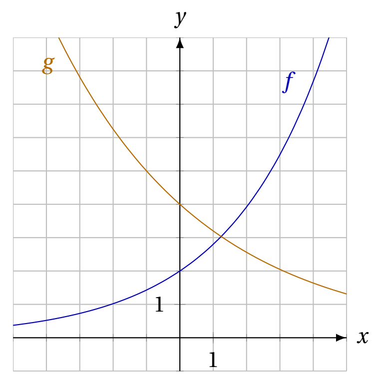

Figure 4.1: The graphs of the two exponential functions $f(x) = 2 \cdot 1.4^x$ and $g(x) = 4 \cdot 0.8^x$.

4.1 EXPONENTIAL GROWTH

If we compare the formula for an exponential function $f(x) = b \cdot a^x$ with the formula for compound interest $A_n = P \cdot a^n$, we see that it is actually the same formula (exchange A_n with $f(x)$, n with x, and P with b). We therefore expect the same type of growth.

We therefore expect exponential functions to grow in such a way that we multiply by the growth factor a, every time x increases by 1. This is shown in figure 4.2. We have the following theorem:

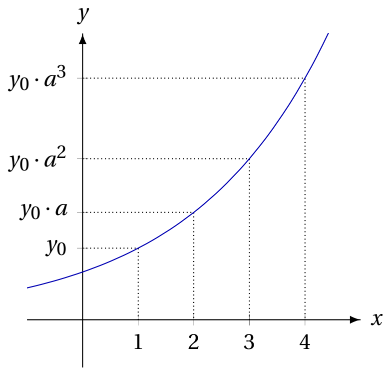

Figure 4.2: Exponential Growth.

Theorem 4.2

Let f be an exponential function. Every time x increases by Δx, the function value of f is multiplied by $a^{\Delta x}$. In other words, when x increases by a fixed amount, the function value increases by a fixed *percentage*.

Proof
If x increases from x_1 to x_2, the function value will increase from

$$y_1 = f(x_1) = b \cdot a^{x_1}$$

to

$$y_2 = f(x_2) = f(x_1 + \Delta x) = b \cdot a^{x_1+\Delta x} = b \cdot a^{x_1} \cdot a^{\Delta x} = y_1 \cdot a^{\Delta x}\,.$$

The new function value y_2 is exactly $y_1 \cdot a^{\Delta x}$, which proves the theorem.■

Table 4.1: Growth of $f(x) = 4 \cdot 2^x$.

x	y
-3	0,5
0	4
3	32
6	256

Example 4.3
In table 4.1, we see an example of exponential growth. Here, the function is $f(x) = 4 \cdot 2^x$, and every time x increases by 3, the function value is multiplied by $2^3 = 8$.

Every time x increases by 1, the function value is multiplied by $a^1 = a$. The size of a therefore determines if an exponential function is increasing or decreasing. We therefore have the following theorem[2]

[2]The theorem follows from the fact that if we multiply any number with a number larger than 1, the result will be larger than the original number. Conversely, if we multiply by a number less than 1, the result will be less than the original.

Theorem 4.4

For an exponential function $f(x) = b \cdot a^x$ we have:

1. If $a > 1$, the function is increasing.
2. If $0 < a < 1$, the function is decreasing.

As we have seen, an exponential function grows by a fixed percentage for a fixed increase in x. Therefore, it makes sense to define the *growth rate*

$$r = a - 1\,.$$

This number is often written as a percentage. For the growth rate, we have the following theorem, which follows from theorem 4.4.

Theorem 4.5

For an exponential function $f(x) = b \cdot a^x$, we have

1. if $r > 0$, the function is increasing,
2. if $r < 0$, the function is decreasing,

where $r = a - 1$ is the *growth rate.*

We may use this to make mathematical models based on a certain growth rate.

Example 4.6
In 2014, the population of Honduras was 8.6 million, and the growth rate was 1.7%.[6] We can therefore describe the population of Honduras using an exponential model with an initial value of 8.6 and a growth rate of 1.7%.

Since the growth rate is 1.7%, the growth factor is

$$a = 1 + 1.7\% = 1 + 0.017 = 1.017\,.$$

Thus the population is given by the function

$$f(x) = 8.6 \cdot 1.017^x ,$$

where x is the number of years since 2014, and $f(x)$ is the population (in millions).

Example 4.7
A bacterial culture grows exponentially such that the number of bacteria can be described by the function

$$B(t) = 364 \cdot 1.72^t ,$$

where t is the time in hours, and $B(t)$ is the number of bacteria.

From this formula, we see that at the time $t = 0$ (initially) there are 364 bacteria. The growth factor is 1.72, i.e. the growth rate is

$$r = 1.72 - 1 = 0.72 = 72\% .$$

Therefore the number of bacteria grows by 72% per hour.

4.2 CALCULATING THE FORMULA

If we know two points on the graph of an exponential function $f(x) = b \cdot a^x$, we are able to calculate the constants a and b in the formula (see figure 4.3).

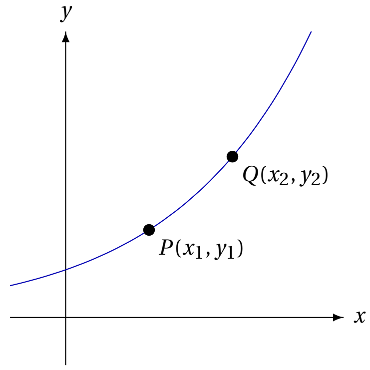

Figure 4.3: The graph of an exponential function passes through the points P and Q.

Theorem 4.8

If the graph of an exponential function $f(x) = b \cdot a^x$ passes through the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then

$$a = \sqrt[x_2 - x_1]{\frac{y_2}{y_1}} \qquad \text{and} \qquad b = \frac{y_1}{a^{x_1}} .$$

Proof
If $P(x_1, y_1)$ is on the graph $f(x) = b \cdot a^x$, then

$$y_1 = b \cdot a^{x_1} . \tag{4.1}$$

Since $Q(x_2, y_2)$ is also on the graph of f, we have

$$y_2 = b \cdot a^{x_2} . \tag{4.2}$$

If we divide equation(4.2) by equation (4.1), we get

$$\frac{y_2}{y_1} = \frac{b \cdot a^{x_2}}{b \cdot a^{x_1}} \qquad \Leftrightarrow$$
$$\frac{y_2}{y_1} = \frac{a^{x_2}}{a^{x_1}} \qquad \Leftrightarrow$$
$$\frac{y_2}{y_1} = a^{x_2 - x_1} \qquad \Leftrightarrow$$

$$\sqrt[x_2-x_1]{\frac{y_2}{y_1}} = a\,,$$

which proves the formula for a.

To prove the formula for b, we solve equation (4.1) for b, and get

$$y_1 = b\cdot a^{x_1} \qquad \Leftrightarrow \qquad \frac{y_1}{a^{x_1}} = b\,.$$

This proves the formula for b. ■

Example 4.9
If the graph of an exponential function $f(x) = b\cdot a^x$ passes through the two points $P(2,12)$ and $Q(5,96)$, we have

$$x_1 = 2, \quad y_1 = 12, \quad x_2 = 5 \quad \text{og} \quad y_2 = 96\,.$$

Now we use the formulas from theorem 4.8 to get

$$a = \sqrt[x_2-x_1]{\frac{y_2}{y_1}} = \sqrt[5-2]{\frac{96}{12}} = \sqrt[3]{8} = 2\,,$$
$$b = \frac{y_1}{a^{x_1}} = \frac{12}{2^2} = \frac{12}{4} = 3\,.$$

The formula of the function is therefore $f(x) = 3\cdot 2^x$.

4.3 DOUBLING TIME AND HALF LIFE

According to theorem 4.2, an exponential function increases by a fixed percentage if x increases by a fixed value. If we look at an exponentially increasing function, it is therefore meaningful to investigate, exactly how much x should increase for the function value to increase by 100%, i.e. to double. This number is known as the *doubling time* T_2.

The function value is multiplied by $a^{\Delta x}$ when x increases by Δx. To determine T_2, we must therefore find a value of Δx, such that $a^{\Delta x} = 2$, since doubling amounts to multiplying by 2. Thus T_2 is the solution to the equation

$$a^{T_2} = 2\,.$$

The solution to this equation is[3]

$$T_2 = \frac{\log(2)}{\log(a)}\,.$$

[3]Remember that the equation $a^x = c$ ha- has the solution $x = \frac{\log(c)}{\log(a)}$.

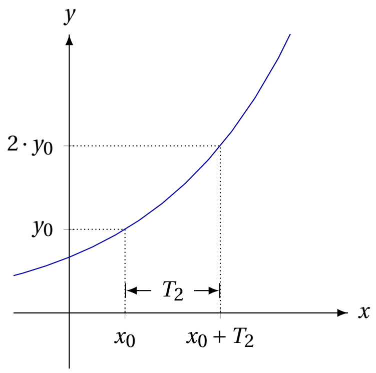

Figure 4.4: When x increases by T_2, the function value is doubled.

Figure 4.4 illustrates the meaning of the doubling time: *Every time x increases by T_2, the function value is doubled.* This applies only to exponential functions.

The concept of doubling time makes no sense when we look at decreasing exponential functions. Here we instead define the *half life*. The half life $T_{\frac{1}{2}}$ is defined analogously to the doubling time. We now have

Theorem 4.10

For an exponential function $f(x) = b \cdot a^x$, we have

1. If f is increasing, the doubling time is $T_2 = \frac{\log(2)}{\log(a)}$.
2. If f is decreasing, the half life is $T_{\frac{1}{2}} = \frac{\log(\frac{1}{2})}{\log(a)}$.

Example 4.11
The exponential function $f(x) = 3 \cdot 1.7^x$ has a growth factor of $a = 1.7$. Therefore the doubling time is

$$T_2 = \frac{\log(2)}{\log(a)} = \frac{\log(2)}{\log(1.7)} = 1.31\,.$$

I.e. every time x increases by 1.31, the function value is doubled.

An increase from $x = 5$ to 6.31 will therefore double the function value, and so will an increase from $x = 100$ to $x = 101.31$.

4.4 EXPONENTIAL REGRESSION

Exponential regression is a method for finding the exponential function, whose graph is closest to a certain series of data points.

This method is built into most spreadsheets and CAS tools. We input the data points and obtain the formula for the exponential function, whose graph best describes the behaviour of our data set. The equation in figure 4.5 is found in this way.

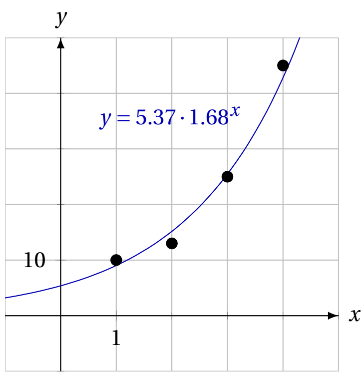

Figure 4.5: An exponential function found using exponential regression.

Numbers and Arithmetic A

Numbers are one of the corner stones of mathematics. Numbers are used for many different purposes: For counting and measuring or to calculate profit and debt.

In the next few sections, we look at different types of numbers and the four arithmetical operations: *addition*, *subtraction*, *multiplication* and *division*.

The first numbers we learn are the numbers we call *natural numbers*. These are the numbers used for counting, i.e.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \dots .$$

If we want to express the notion of nothing, we use the number 0. However, we do not place this among the natural numbers.

A.1 ADDITION

The simplest of the four arithmetical operations is addition. This operation does expresses the same idea as the word "and".[1]

[1] The symbol + in all likelihood comes from the word *et* which means "and" in latin.[1]

E.g. if we have a pile of 7 items and a pile of 11 items, we have a total of "7 items and 11 items", i.e. 18 items. In mathematics we write

$$7 + 11 = 18 .$$

The two numbers 7 and 11 are called *terms* and the result 18 is called a *sum*.

Since the sign + tells us to add the values, the order cannot matter. Therefore we expect that

$$7 + 11 = 11 + 7 ,$$

which is true.

A.2 SUBTRACTION AND NEGATIVE NUMBERS

Using the natural numbers, we cannot express the idea of loss or debt. To do this, we need the negative numbers.[2] We begin by looking at negative

[2] The negative numbers are actually quite a new invention in mathematics. Well into the 18th century, some mathematicians believed that negative numbers did not actually exist.[4]

whole numbers:

$$-1, -2, -3, -4, -5, -6, \dots\,.$$

In a way, every negative number is an "opposite number" of a positive number. In mathematics we call this an *inverse* number (with respect to addition). Adding a number and its inverse yields e.g. [3]

$$3 + (-3) = 0 \quad \text{og} \quad (-3) + 3 = 0\,.$$

[3]In the calculation, we write parentheses around the number −3. We do this to show that − is a sign, i.e. it belongs to the number −3. Generally, we may not write a sign without parentheses directly after an arithmetical operation.

We may argue that $-(-3)$ must be the inverse of -3 and that we therefore must have

$$-(-3) + (-3) = 0\,.$$

But because $3 + (-3) = 0$, we must also have

$$-(-3) = 3\,.$$

This applies to all numbers (not just the number 3).

Using the negative numbers, we may *define* subtraction as adding the inverse number. An example might be

$$8 - 2 = 8 + (-2)\,.$$

This also explains why the numbers are not directly interchangeable. $2-8$ is not the same as $8-2$ because the sign $-$ actually belongs to the number 2. When we add, however, the order does not matter. Thus we can do as follows

$$8 - 2 = 8 + (-2) = -2 + 8\,.$$

Here we see that the sign $-$ is always before the number 2.

The two numbers 8 and 2 in the calculation are called *terms* (as with addition) while the result (6) is called a *difference*.

A.3 MULTIPLICATION

Multiplication may be viewed as an extension of addition since e.g.

$$7 \cdot 4 = \overbrace{4+4+4+4+4+4+4}^{7\text{ times}} = 28\,.$$

Here, we see why the symbol "·" is called "times".

For *addition*, we may easily argue that the order does not matter. This is also the case for multiplication, though it may not be as obvious why e.g. $7 \cdot 4 = 4 \cdot 7$. However, if we view the number $4 \cdot 7$ as the area of a rectangle where one side is 4 and the other 7, exchanging the numbers 4 and 7 only amounts to rotating the rectangle—hence the area stays the same. This is illustrated in figure A.1.

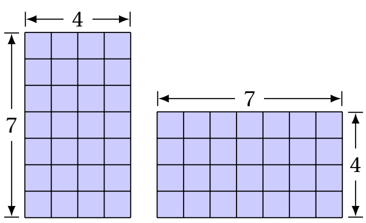

Figure A.1: From this figure, we may argue that $7 \cdot 4 = 4 \cdot 7$.

The numbers that are multiplied (7 and 4) are called *factors* and the result (28) is called the *product* of the two numbers.

Sign

It is easy to show what happens when we multiply positive numbers. But what happens if there are negative numbers involved?

Negative numbers may be interpreted as debt. In this light, a calculation like $-6 \cdot 3$ should be interpreted as a debt of 6 items multiplied by 3, i.e. a debt of 18 items. Thus we have

$$-6 \cdot 3 = -18\,.$$

Since the order of multiplication does not matter, we must also have[4]

$$3 \cdot (-6) = -18\,.$$

[4] Notice the parenthesis around -6. We write this to indicate that the sign belongs to the number 6 (and it is not optional).

From this, we see that multiplying a positive by a negative number yields a negative number.

But what happens when we multiply two negative numbers? Here, we can use the following argument: $-2 \cdot (-4)$ is the (additive) inverse of $2 \cdot (-4)$. And because $2 \cdot (-4) = -8$, we have

$$-2 \cdot (-4) = -(-8) = 8\,.$$

Hence, when we multiply two negative numbers, the result is positive. The sign rules for multiplication are listed in table A.1.

Table A.1: Sign rules for multiplication.

Rule	Example
$(+) \cdot (+) = (+)$	$2 \cdot 3 = 6$
$(+) \cdot (-) = (-)$	$2 \cdot (-4) = -8$
$(-) \cdot (+) = (-)$	$(-3) \cdot 5 = -15$
$(-) \cdot (-) = (+)$	$(-4) \cdot (-2) = 8$

A.4 DIVISION

If 2 people share 6 items, they get 3 each. The calculation we perform is a division:

$$\frac{6}{2} = 3.$$

The number which we divide (6) is called the *divident*, and the number we divide by (2) is called the *divisor*. The result of a division is called a *quotient*.

Division is the opposite of multiplication. The number we find in the calculation above is the answer to the question: *Which number do we multiply by 2 to get 6?*[5]

[5] This explains why we cannot divide by 0. The result of the calculation $\frac{4}{0}$ answers the question: *What do we multiply by 0 to get 4?* But a number multiplied by 0 is always 0, i.e. a number multiplied by 0 can never be 4. Therefore it makes no sense to divide by 0.

The result of a division is not necessarily a whole number. We therefore have to introduce some other numbers, namely fractions, i.e. numbers like $\frac{1}{2}$, $\frac{5}{3}$ and $-\frac{7}{13}$.

Fractions are not as easy to work with as whole numbers; how do we add, e.g., $\frac{1}{2}$ and $\frac{2}{3}$? How we calculate with fractions can be deduced from what is written above. However, there are quite a few things to deduce, so this will be explained in a later section.

Sign

Division can be viewed as a form of multiplication. The calculation $\frac{6}{2} = 3$ may also be written as

$$6 \cdot \frac{1}{2} = 3\,.$$

But if every division can be turned into a multiplication, the same sign rules must apply.

Therefore e.g.

$$\frac{-20}{5} = -4, \qquad \frac{14}{-2} = -7 \qquad \text{og} \qquad \frac{-18}{-6} = 3\,.$$

The sign rules for division are listed in table A.2.

Table A.2: Sign rules for division.

Rule	Example
$\frac{(+)}{(+)} = (+)$	$\frac{6}{2} = 3$
$\frac{(+)}{(-)} = (-)$	$\frac{10}{-5} = -2$
$\frac{(-)}{(+)} = (-)$	$\frac{-14}{2} = -7$
$\frac{(-)}{(-)} = (+)$	$\frac{-18}{-3} = 6$

Decimal Numbers

Numbers that are not whole are often written as decimal numbers instead of fractions. En example of a decimal is the number 1.472.

In principle a decimal number is a sum of fractions, e.g.

$$1{,}472 = 1 + \frac{4}{10} + \frac{7}{100} + \frac{2}{1000}\,.$$

However, this is not something we have to think about when we use decimals.

Actually, every number may be written as a decimal number. We have

$$\frac{1}{2} = 0.5$$
$$\frac{1}{3} = 0{,}333333333\ldots$$
$$\frac{10}{7} = 1.42857\,142857\,142857\ldots$$

As we see from the last two numbers, we sometimes need an infinite amount of decimals to write a number as a decimal number. For this reason, a fraction is more precise than a decimal number.[6]

[6] This also applies to $\frac{1}{2}$. Even though this can be written as 0.5, we are more precise when we write $\frac{1}{2}$. If we write 0.5, it is impossible for a reader to see if the number actually has an infinite amount of zeroes after the 5, or if it is the result of a rounding from e.g. 0.496.

What we also see from the decimal numbers above is that even though e.g. $\frac{10}{7}$ cannot be written precisely as a decimal number, there is a recurring pattern in the decimals. We write the same sequence of numbers over and over. This is true for every fraction when we write it as a decimal number; if a fraction is written as a decimal number it either has a finite amount of decimals or the decimals while repeat the same pattern ad infitum. We call these numbers *rational numbers*, i.e. numbers that can be written as a *ratio* of two whole numbers.

Irrational Numbers

If we write a number as a decimal, we either get a finite amount of decimals or an endlessly repeating pattern. From this follows that a decimal number without any pattern in the decimals cannot be written as a fraction.

But is it even possible to conceive a number, which cannot be written as a fraction? It turns out that there are infinitely many of such numbers, a well-known example is the number π—the ratio of the circumference and the diameter of a circle. To 20 decimal places, we have

$$\pi = 3{,}14159265358979323846\ldots\,.$$

Here there is no pattern in the decimals, and there never will be, no matter how many decimals we calculate.

These numbers, which cannot be written as fractions, are called *irrational numbers*. The rational and the irrational numbers collectively make up the *real numbers*. If we view numbers as points on a number line, every point on the line corresponds to a real number.

We can now group numbers into different sets:

The natural numbers, which we use for counting: $1, 2, 3, 4, \ldots$.

The whole numbers, including the negatives: $\ldots, -3, -2, -1, 0, 1, 2, \ldots$.

The rational numbers, which are numbers that may be written as fractions.[7]

The real numbers, i.e. *all* numbers.

[7] The whole numbers are also rational because every whole number can be written as a fraction; e.g. $4 = \frac{8}{2}$ and $-5 = \frac{-15}{3}$.

A.5 POWERS AND ROOTS

$4+4+4$ can be written as $4 \cdot 3$. Similarly, a shorthand exists for e.g. $5 \cdot 5 \cdot 5 \cdot 5$. Since we multiply four 5's, we instead write 5^4. I.e.[8]

$$5^4 = \overbrace{5 \cdot 5 \cdot 5 \cdot 5}^{4 \text{ times}}.$$

[8] In 5^4, the number 5 is called the *base* and the number 4 is called the *exponent*.

5^4 is called "5 raised to the power 4" or "the 4th power of 5".

The opposite calculation is called a *root* of number. E.g. we may calculateNotice that it is not called the "2nd root" but the *square root* and we do not write the number 2. I.e. we write $\sqrt{49}$ and not $\sqrt[2]{49}$.

$$\begin{aligned} \sqrt[4]{81} &\quad \text{the 4th root of 81,} \\ \sqrt[3]{125} &\quad \text{the cube root of 125,} \\ \sqrt[5]{32} &\quad \text{the 5th root of 32,} \\ \sqrt{49} &\quad \text{the square root of 49.} \end{aligned}$$

The results of these calculations are

$$\begin{aligned} \sqrt[4]{81} = 3 &\qquad \text{because } 3^4 = 81 \\ \sqrt[3]{125} = 5 &\qquad \text{because } 5^3 = 125 \\ \sqrt[5]{32} = 2 &\qquad \text{because } 2^5 = 32 \\ \sqrt{49} = 7 &\qquad \text{because } 7^2 = 49\,. \end{aligned}$$

There are a lot of rules we can use when we do calculations with powers and roots; these are described in a later chapter.

A.6 ORDER OF OPERATIONS

When we calculate e.g. $7 + 5 \cdot 3^2$, we need to know in which order to do the different steps. Do we add first 7 and 5 or do we calculate first 3^2?

Therefore we have rules that describe the order, in which we calculate. This ensures that everybody gets the same (correct) result from a certain calculation.[9]

[9] The order of operations is the one that makes sense logically. We multiply before we add because $4 \cdot 3 = 4+4+4$. Therefore

$$2+4\cdot 3 = 2+4+4+4\,,$$

and we therefore have to multiply first (unless we want to rewrite every multiplication as an addition).

Theorem A.1: The order of operations

When we do a calculation, the arithmetical operations are performed in the following order:

1. First we calculate powers and roots.
2. Then we multiply and divide.
3. And lastly we add and subtract.

This order can only be changed using parentheses. If some part of a calculation is in a parenthesis, we view this as a separate calculation, which must be done *first*.

Some examples of calculations are

Example A.2
Using the order of operations:

$$\begin{aligned} 2\cdot 17 - 4\cdot 2^3 &= 2\cdot 17 - 4\cdot 8 && \text{First we calculate } 2^3. \\ &= 34 - 32 && \text{Then we multiply.} \\ &= 2 && \text{And subtract.} \end{aligned}$$

Example A.3
An example including parentheses:

$$\begin{aligned} (6+2)\cdot 5 + 3\cdot\sqrt{16} &= 8\cdot 5 + 3\cdot\sqrt{16} && \text{The parenthesis is calculated first.} \\ &= 8\cdot 5 + 3\cdot 4 && \text{Then the roots.} \\ &= 40 + 12 && \text{Then we multiply.} \\ &= 52 && \text{And lastly, we add.} \end{aligned}$$

Hidden Parentheses

Some calculations actually include parentheses, which are not obviously there.

In a calculation such as $\frac{3+17}{10}$, we have to calculate $3+17$ first—before we divide. When we write a division, there are actually parentheses around both the numerator and the denominator, and this must be taken into account when we calculate.

The same applies to roots. In e.g. $\sqrt{17-8}$, we must calculate $17-8$ before we apply the square root.

Example A.4
In the following examples, it is shown explicitly where the hidden parentheses are:

$$\frac{3+9}{4} = \frac{(3+9)}{4}$$

$$
\begin{aligned}
\frac{50}{7-2} &= \frac{50}{(7-2)} \\
7^{2+1} &= 7^{(2+1)} \\
\sqrt{7+9} &= \sqrt{(7+9)} \\
\sqrt[5-2]{8} &= \sqrt[(5-2)]{8}\,.
\end{aligned}
$$

These are important to remember.

The example above does not cover every possible case. But as a rule, if something looks like a separate part of a calculation, it probably is.

Fractions

B

A fraction is a number of equal parts of a whole. We write fractions as two whole numbers above and below a straight line:

$$\frac{2}{3}, \quad \frac{7}{4}, \quad \frac{-13}{29}.$$

The top number is called the *numerator*, and the bottom number is called the *denominator*.[1]

[1] The numerator and the denominator are always whole numbers, possibly negative. However, the denominator cannot be 0.

The fraction $\frac{2}{3}$ is the number we get when we divide 1 whole into 3 parts and take 2 of them. A fraction may also be interpreted as the exact result of dividing the numerator by the denominator.

If we want to visualise a fraction, we may do so by using the number line (see figure B.1).

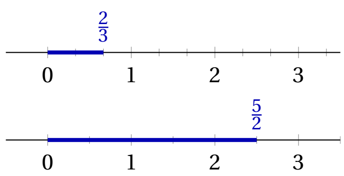

Figure B.1: The two numbers $\frac{2}{3}$ and $\frac{5}{2}$ on the number line.

B.1 EQUIVALENT FRACTIONS

If we interpret a fraction as the result of dividing the numerator by the denominator, a fraction may actually be equal to a whole number. This is true if the denominator divides the numerator,[2]

[2] Alternatively, whole numbers may be seen as fractions with denominator 1, e.g. $8 = \frac{8}{1}$.

$$\frac{10}{5} = 2, \qquad \frac{36}{9} = 4, \qquad \frac{-27}{3} = -9.$$

Even if this is not the case, we can sometimes write the fraction with a smaller numerator and denominator (*simplifying* the fraction). This is possibly if a whole number exists, which divides both the numerator and the denominator.

This situation is illustrated in figure B.2. Here we see that $\frac{4}{6} = \frac{2}{3}$. In the fraction $\frac{4}{6}$, the number 2 divides the numerator and the denominator. Since a fraction is, in a way, a ratio between the numerator and the denominator, its size does not change if we divide the numerator and the denominator by the same number. We therefore have

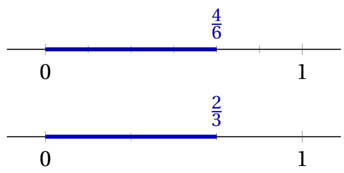

Figure B.2: From the two number lines, we see that $\frac{4}{6} = \frac{2}{3}$.

$$\frac{4}{6} = \frac{4/2}{6/2} = \frac{2}{3}.$$

This does not change the value of the fraction. The number is exactly the same as before, we just write it using smaller numbers for the numerator and the denominator—and it is always easier to work with smaller numbers.

Example B.1
A few examples of this kind of reduction are:

$$\frac{15}{36} = \frac{15/3}{36/3} = \frac{5}{12}\,,$$
$$\frac{24}{56} = \frac{24/8}{56/8} = \frac{3}{7}\,,$$
$$\frac{27}{18} = \frac{27/9}{18/9} = \frac{3}{2}\,.$$

If dividing by the same number in the numerator and the denominator does not change the value of the fraction, we may also multiply.[3] If we do this, the numerator and the denominator become larger, and this does not seem very useful. But it turns out to useful indeed when we need to add fractions—see the section below.

[3]It is important to remember that we have to divide or multiply by the *same number* in the numerator and the denominator. Otherwise we change the value of the fraction.

Example B.2
If we multiply $\frac{3}{4}$ by 5 in the numerator and the denominator, we get

$$\frac{3}{4} = \frac{3\cdot 5}{4\cdot 5} = \frac{15}{20}\,.$$

Multiplying $\frac{8}{5}$ by $\frac{3}{3}$ yields

$$\frac{8}{5} = \frac{8\cdot 3}{5\cdot 3} = \frac{24}{15}\,.$$

B.2 ADDITION AND SUBTRACTION

It turns out that we can only add fractions if they have the same denominator. If this is the case, we just add the numerators. E.g.

$$\frac{2}{5} + \frac{4}{5} = \frac{2+4}{5} = \frac{6}{5}\,.$$

This is illustrated in figure B.3.

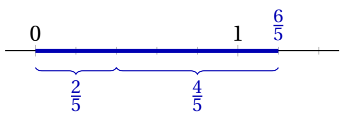

Figure B.3: The calculation $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$ illustrated on the number line.

It is not possible to add fractions with different denominators. Nonetheless, we would like to have method for adding e.g. $\frac{1}{4}$ and $\frac{2}{3}$. If we can only add fractions when they have the same denominator, we need a method for producing equal denominators.

We do this by using the method described in the above section. We multiply the fraction $\frac{1}{4}$ by 3 in the numerator and the denominator and multiply $\frac{2}{3}$ by 4. We then get

$$\frac{1}{4} + \frac{2}{3} = \frac{1\cdot 3}{4\cdot 3} + \frac{2\cdot 4}{3\cdot 4} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}\,.$$

Here both fractions end up with the numerator 12; they can then be added.

In this calculation we multiply the first fraction with the denominator of the second, and vice versa. This always works.[4]

[4]But it is not always necessary. Sometimes smaller numbers exist which also yield a common denominator.

Example B.3
A few examples of additions involving fractions:

$$\frac{1}{3}+\frac{1}{2}=\frac{1\cdot 2}{3\cdot 2}+\frac{1\cdot 3}{2\cdot 3}=\frac{2}{6}+\frac{3}{6}=\frac{5}{6}\,,$$
$$\frac{7}{4}+\frac{2}{11}=\frac{7\cdot 11}{4\cdot 11}+\frac{2\cdot 4}{11\cdot 4}=\frac{77}{44}+\frac{8}{44}=\frac{85}{44}\,,$$
$$\frac{3}{5}+\frac{1}{10}=\frac{3\cdot 2}{5\cdot 2}+\frac{1}{10}=\frac{6}{10}+\frac{1}{10}=\frac{7}{10}\,.$$

In the last calculation, we see that the common denominator is not the product of the two original denominators, but instead the smallest number, which both denominators divide.[5]

[5]Usually, we try to calculate the result using the smallest numbers possible—simply because it is easier.

If we want to subtract fractions, we can use the exact same method. We can only subtract fractions if they have a common denominator. E.g.

$$\frac{8}{13}-\frac{3}{13}=\frac{8-3}{13}=\frac{5}{13}\,.$$

If the two fractions do not have a common denominator, we multiply the numerator and denominator of each fraction to give them a common denominator.

Example B.4
Three examples of subtracting fractions:

$$\frac{4}{3}-\frac{2}{5}=\frac{4\cdot 5}{3\cdot 5}-\frac{2\cdot 3}{5\cdot 3}=\frac{20}{15}-\frac{6}{15}=\frac{14}{15}\,,$$
$$\frac{7}{8}-\frac{1}{2}=\frac{7}{8}-\frac{1\cdot 4}{2\cdot 4}=\frac{7}{8}-\frac{4}{8}=\frac{3}{8}\,,$$
$$\frac{2}{3}-\frac{4}{5}=\frac{2\cdot 5}{3\cdot 5}-\frac{4\cdot 3}{5\cdot 3}=\frac{10}{15}-\frac{12}{15}=\frac{-2}{15}\,.$$

It is, by the way, good practice to simplify the result as much as possible.

B.3 MULTIPLICATION AND DIVISION

The area of a rectangle is the product of its length and width. Therefore the product of two numbers can be interpreted as the area of the corresponding rectangle. From this we see that $\frac{4}{5}\cdot\frac{2}{3}$ must be the area of a rectangle, whose length is $\frac{4}{5}$ and whose width is $\frac{2}{3}$. Figure B.4 shows the result of this multiplication; we have

$$\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}\,.$$

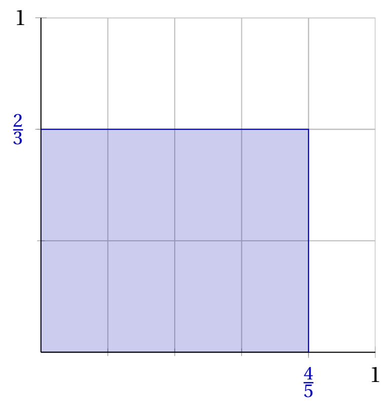

Figure B.4: Here we see that $\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}$.

Analysing the figure, we see that the number of coloured rectangles (the numerator of the result) is the product of the two numerators (4 and 2). The total number of small rectangles, which make up the area 1, can be found by multiplying the two denominators (5 and 3). This means that we have

$$\frac{4}{5}\cdot\frac{2}{3}=\frac{4\cdot 2}{5\cdot 3}=\frac{8}{15}\,.$$

From this argument we see that we multiply fractions by multiplying their numerators and their denominators.

Example B.5

A few multiplications involving fractions:

$$\begin{aligned}\frac{3}{2}\cdot\frac{5}{7}&=\frac{3\cdot 5}{2\cdot 7}=\frac{15}{14}\,,\\ \frac{4}{9}\cdot\frac{7}{11}&=\frac{4\cdot 7}{9\cdot 11}=\frac{28}{99}\,,\\ \frac{1}{3}\cdot\frac{5}{2}\cdot\frac{13}{7}&=\frac{1\cdot 5\cdot 13}{3\cdot 2\cdot 7}=\frac{65}{42}\,.\end{aligned}$$

The final arithmetical operation is division. We use the following calculation as an example:

$$\frac{4}{7}\Big/\frac{2}{5}\,.$$

Before we analyse this calculation, we first note that [6]

$$\frac{2}{5}\cdot\frac{5}{2}=\frac{2\cdot 5}{5\cdot 2}=\frac{10}{10}=1\,.$$

[6] The fraction we get by exchanging the numerator and the denominator is called the *reciprocal* of the original fraction.

Now, if we multiply the original calculation $\frac{4}{7}\big/\frac{2}{5}$ by 1, we do not change the result. Therefore we write

$$\frac{4}{7}\Big/\frac{2}{5}\cdot 1\,.$$

But since $\frac{2}{5}\cdot\frac{5}{2}=1$, we might as well write

$$\frac{4}{7}\Big/\frac{2}{5}\cdot\left(\frac{2}{5}\cdot\frac{5}{2}\right)\,.$$

The order in which we multiply and divide does not matter. We can therefore move the parenthesis and write

$$\left(\frac{4}{7}\Big/\frac{2}{5}\cdot\frac{2}{5}\right)\cdot\frac{5}{2}\,.$$

Inside the parenthesis, we now have $\frac{4}{7}$ divided by $\frac{2}{5}$ and then multiplied by $\frac{2}{5}$. Since division and multiplication are opposite operations, this does not change the number $\frac{4}{7}$. Therefore we might as well remove it and just write

$$\left(\frac{4}{7}\right)\cdot\frac{5}{2}\,,$$

which is the same $\frac{4}{7}\cdot\frac{5}{2}$.

All the way through this argument, we have looked at the same calculation. We therefore conclude that

$$\frac{4}{7}\Big/\frac{2}{5}=\frac{4}{7}\cdot\frac{5}{2}\,.$$

Thus we may change a division by a fraction into a multiplication by the reciprocal fraction.

Example B.6
A few examples:

$$\frac{3}{8}\Big/\frac{7}{5} = \frac{3}{8}\cdot\frac{5}{7} = \frac{15}{56}\,,$$
$$\frac{1}{2}\Big/\frac{11}{5} = \frac{1}{2}\cdot\frac{5}{11} = \frac{5}{22}\,,$$
$$\frac{7}{6}\Big/\frac{3}{13} = \frac{7}{6}\cdot\frac{13}{3} = \frac{91}{18}\,.$$

B.4 FRACTIONS AND WHOLE NUMBERS

If a calculation involves whole numbers as well as fractions, the easiest way to proceed is to write the whole numbers as fractions (with denominator 1), e.g.

$$8 = \frac{8}{1}, \qquad 12 = \frac{12}{1} \qquad \text{og} \qquad -3 = \frac{-3}{1}\,.$$

Now the calculations only involve fractions, and we may use the methods described above.

Example B.7
In this example, we look at the calculation $2 + \frac{3}{4}$. If we write the number 2 as a fraction, we get

$$\frac{2}{1} + \frac{3}{4}\,.$$

Now we need a common denominator, so we multiply the numerator and the denominator of the first fraction by 4:

$$\frac{2}{1} + \frac{3}{4} = \frac{2\cdot 4}{1\cdot 4} + \frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}\,.$$

The result of this addition is $\frac{11}{4}$.

Example B.8
The result of the division $\frac{4}{3}/5$ can be found by writing the number 5 as the fraction $\frac{5}{1}$. Then we have

$$\frac{4}{3}\Big/\frac{5}{1} = \frac{4}{3}\cdot\frac{1}{5} = \frac{4}{15}\,.$$

B.5 SIGN

Calculations involving fractions with negative numbers in the numerator and/or the denominator are done in exactly the same way as divisions since a fractions is, in essence, a form of division.

If we divide a positive number by a negative number, or vice versa, the result is negative. Therefore we have

$$\frac{-6}{11} = \frac{6}{-11}\,.$$

Usually we write the sign outside the fraction, i.e.

$$-\frac{6}{11}\,.$$

A negative sign outside the fraction means that the fraction itself is negative. Since we get the same result if the negative sign is written on the numerator or the denominator, it does not matter where we write it (as long as there is *only one* negative sign).

If both the numerator and the denominator have a negative sign, the resulting fraction is actually positive, i.e.

$$\frac{-13}{-7} = \frac{13}{7}\,.$$

If we are presented with a calculation involving numerous multiplications and divisions, we can determine the sign of the result by remembering that every pair of negative signs "vanishes". If the amount of negative signs is even, the result is therefore positive; if the amount of negative signs is odd, the result is negative.

Powers and Roots

C

In this chapter, we look at some of the rules for powers and roots. First, we look at powers where the exponent is a whole number. This is then expanded to exponents, which are negative numbers or fractions.

C.1 INTEGER EXPONENTS

Raising a number to a power is defined in the following way:[1]

$$3^5 = \overbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}^{5 \text{ times}} .$$

Roots are the opposite operation, e.g.

$$\sqrt[4]{16} = 2, \qquad \text{because } 2^4 = 16 .$$

If we multiply two powers that have the same base, we may do this

$$7^2 \cdot 7^4 = \overbrace{\underbrace{7 \cdot 7}_{2 \text{ times}} \cdot \underbrace{7 \cdot 7 \cdot 7 \cdot 7}_{4 \text{ times}}}^{2+4=6 \text{ times}} = 7^6 .$$

Dividing two powers that have the same base yields this:

$$\frac{4^5}{4^3} = \frac{4 \cdot 4 \cdot \cancel{4} \cdot \cancel{4} \cdot \cancel{4}}{\cancel{4} \cdot \cancel{4} \cdot \cancel{4}} = 4 \cdot 4 = 4^2 ,$$

i.e.

$$\frac{4^5}{4^3} = 4^{5-3} .$$

If we have two consecutive powers we get

$$(2^4)^3 = \overbrace{\underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4 \text{ times}} \cdot \underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4 \text{ times}} \cdot \underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4 \text{ times}}}^{4 \cdot 3 \text{ times}} = 2^{4 \cdot 3} = 2^{12} .$$

Now we have seen what happens when we combine powers with the same base. Next, we examine what happens when we combine powers with the same exponent.

[1] In the calculation 3^5, we call the number 3 the *base* and the number 5 the *exponent*.

Multiplying two powers with the same exponent yields e.g.

$$5^3 \cdot 2^3 = 5 \cdot 5 \cdot 5 \cdot 2 \cdot 2 \cdot 2 = 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 = (5 \cdot 2) \cdot (5 \cdot 2) \cdot (5 \cdot 2) = (5 \cdot 2)^3 .$$

Division leads to

$$\frac{7^4}{3^4} = \frac{7 \cdot 7 \cdot 7 \cdot 7}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{7}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} = \left(\frac{7}{3}\right)^4 .$$

All of these calculations may be generalised. Thus we obtain these 5 power rules:

Theorem C.1

If m and n are two natural numbers, and a and b are two arbitrary numbers, we have

1. $a^m \cdot a^n = a^{m+n}$.
2. If a is not 0, and $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$.
3. $(a^m)^n = a^{m \cdot n}$.
4. $a^n \cdot b^n = (a \cdot b)^n$.
5. If b is not 0, then $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$.

C.2 RATIONAL EXPONENTS

In this section, we look at exponents that are not positive integers. This raises an interesting question: We know what 5^4 means, but how do we interpret e.g. 2^{-7} or $3^{\frac{1}{4}}$?

We assign a meaning to calculations such as these by demanding that the rules in theorem C.1 must be true, no matter which values we assign to the exponents. It turns out that it is only possible for the rules to be true, if we define powers with negative or fractional exponents in a certain way.

If, e.g., we calculate 5^0, rule 2 in theorem C.1 yields

$$5^0 = 5^{2-2} = \frac{5^2}{5^2} = 1 .$$

The base here is 5, but it could have been any number. From a similar calculation we can just as easily show that $7^0 = 1$. We can therefore generalise this to any number.[2]

If the exponent is negative, we can use the same rule to obtain[3]

$$6^{-3} = 6^{0-3} = \frac{6^0}{6^3} = \frac{1}{6^3} .$$

This calculation can also be performed using other numbers. Thus we find e.g. $13^{-7} = \frac{1}{13^7}$. The argument works for any base but 0.

If the exponent is a fraction, we use rule 3 in theorem C.1 to calculate e.g. $(8^{\frac{1}{3}})^3$. This yields

$$(8^{\frac{1}{3}})^3 = 8^{\frac{1}{3} \cdot 3} = 8^1 = 8 .$$

[2] Except 0—because we cannot divide by 0.

[3] Here we use that we have just shown that $5^0 = 1$, $6^0 = 1$, $43^0 = 1$, etc.

But we also know that[4]

$$(\sqrt[3]{8})^3 = 8\,.$$

Therefor

$$8^{\frac{1}{3}} = \sqrt[3]{8}\,.$$

This explains how to interpret the calculation if the exponent is $\frac{1}{2}$, $\frac{1}{7}$ or $\frac{1}{73}$; but it does not tell us what to do if the numerator of the fraction is not 1.

Here we instead look at the calculation

$$4^{\frac{5}{7}} = 4^{5\cdot\frac{1}{7}} = (4^5)^{\frac{1}{7}} = \sqrt[7]{4^5}\,.$$

If the power rules must apply to all numbers, we need the following definition:

Definition C.2

1. $a^0 = 1$ (if $a \neq 0$).
2. $a^{-n} = \frac{1}{a^n}$ (if $a \neq 0$).
3. $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

For rational exponents, we have the same rules as for integer exponents,[5] so we have the following theorem:

Theorem C.3

We have the following rules

1. $a^x \cdot a^y = a^{x+y}$.
2. $(a^x)^y = a^{x\cdot y}$.
3. $\dfrac{a^x}{a^y} = a^{x-y}$.
4. $a^x \cdot b^x = (a\cdot b)^x$.
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$.

Since roots can be calculated by raising to a fractional power, we also have the following theorem:[6]

Theorem C.4

If $a > 0$ and $b > 0$ then

1. $\sqrt[x]{a}\cdot\sqrt[x]{b} = \sqrt[x]{a\cdot b}$.
2. $\dfrac{\sqrt[x]{a}}{\sqrt[x]{b}} = \sqrt[x]{\dfrac{a}{b}}$.

In theorem C.4, there are no rules for the expressions $\sqrt[x]{a}\cdot\sqrt[y]{a}$ og $\frac{\sqrt[x]{a}}{\sqrt[y]{a}}$. This is because in these two cases it is always easier to write the roots as powers before reducing.

If we really want to, it is possible two deduce two formulas. This is left as an exercise for the reader.

[4] Because roots are the opposites of powers.

[5] It was this that led us to the definition C.2.

[6] The two rules follow from e.g.

$$\sqrt[3]{5}\cdot\sqrt[3]{2} = 5^{\frac{1}{3}}\cdot 2^{\frac{1}{3}} = (5\cdot 2)^{\frac{1}{3}} = \sqrt[3]{5\cdot 2}$$

$$\frac{\sqrt[7]{4}}{\sqrt[7]{11}} = \frac{4^{\frac{1}{7}}}{11^{\frac{1}{7}}} = \left(\frac{4}{11}\right)^{\frac{1}{7}} = \sqrt[7]{\frac{4}{11}}\,.$$

Algebra

D

Algebra as a subject is about the rules that apply when we calculate. In mathematics we sometimes need to use numbers, whose values we do not know. These numbers are referred to as *unknowns*. Instead of the unknown number, we write a letter, e.g. x, y, a or A.[1]

[1] It is important to remember that we distinguish between upper and lower case letters–i.e. a and A are not the same number.

If a calculation involves unknowns, we cannot calculate a final result. But we may sometimes be able to reduce or simplify the calculation. This makes the calculation easier when we finally get to know the values of the unknowns.

Since e.g.

$$5+5+5=3\cdot 5\,,$$

we know that

$$x+x+x=3\cdot x\,,$$

no matter which value x has. Similarly, we have

$$8\cdot 8\cdot 8\cdot 8=8^4\,,$$

so

$$x\cdot x\cdot x\cdot x=x^4\,.$$

Hence algebraic rules may be used to simplify calculations and formulas to make them easier to work with.

The algebraic rules we use here are no different from the usual arithmetical rules—i.e. for calculations involving numbers. This is because the letters above *are* numbers. There is, however, one small difference: In algebraic expressions we do not necessarily write the multiplication sign if omitting it does not lead to confusion.[2] Therefore

[2] When calculating $7\cdot 3$, the multiplication sign is necessary–since 73 is not the same as $7\cdot 3$. But we do not need the sign when we write $7x$.

$$\begin{aligned}
4p &= 4\cdot p\\
3xy &= 3\cdot x\cdot y\\
5w^2 &= 5\cdot w^2\\
2y^3z &= 2\cdot y^3\cdot z\\
7ab^2 &= 7\cdot a\cdot b^2\\
2(x+y) &= 2\cdot(x+y)\\
(5-x)(2-x) &= (5-x)\cdot(2-x)\,.
\end{aligned}$$

D.1 LIKE TERMS

If we add x and x, we get $2x$. Therefore the following is also correct:

$$3x + 4x = \overbrace{x + x + x}^{\text{3 terms}} + \overbrace{x + x + x + x}^{\text{4 terms}} = 7x\,.$$

If the letters are the same, we may add the numbers (or subtract them).

Example D.1
A few examples of adding or subtracting like terms.

$$\begin{aligned} 2x + 5x &= 7x \\ 5p^2 + 11p^2 &= 16p^2 \\ 4y + 7y + 2y &= 13y \\ 8xy - 3xy &= 5xy \\ 7w^3 - 15w^3 &= -8w^3\,. \end{aligned}$$

Terms such as $2x$ and $5x$ are called *like terms*. If we add two like terms, we add the numbers.[3] The terms are only like terms if the letters are *exactly* the same. This means we cannot add e.g. $2a$ and $4b$.

[3] If we subtract, we use a similar rule.

Example D.2
Since we can only add like terms, we have

$$\begin{aligned} 3x - 8y + 6x &= 3x + 6x - 8y = 9x - 8y \\ 4w + 7u - w + 5uw &= 4w - w + 7u + 5uw = 3w + 7u + 5uw \\ -3y + 4z + 5y - z &= -3y + 5y + 4z - z = 2y + 3z \\ 4x + 3x^2 + 2x &= 4x + 2x + 3x^2 = 6x + 3x^2\,. \end{aligned}$$

From this example, we see that $4x$ and $3x^2$ are not like terms. This is because the letters have to be *exactly* the same—and x and x^2 are not raised to the same power.[4]

[4] However, $3ab$ and ba are like terms, since the order of multiplication does not matter, i.e. $ba = ab$. The same argument also applies to e.g. xy^2 and y^2x; but not yx^2, since here the wrong number (x instead of y) is squared.

D.2 PARENTHESES

When we simplify expressions, we often use the following 3 rules, which apply to addition and multiplication..

The commutative law: $a + b = b + a$ og $a \cdot b = b \cdot a$.

The associative law: $a + (b + c) = (a + b) + c$ og $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

The distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$

The commutative law merely states that the order of addition or multiplication does not matter. The associative law states that some parentheses are irrelevant, e.g.

$$8x + (3x + 6x) = (8x + 3x) + 6x\,.$$

The parenthesis here is irrelevant. We might as well just write

$$8x + 3x + 6x\,.$$

The sum of these three terms is $17x$.

If a parenthesis is preceded by a "+", we can just remove the parenthesis. This is not true if it is preceded by a "−". Here we need the distributive law to find out how to proceed.

The distributive law follows from the argument sketched in figure D.1, and it tells us how to multiply a number with a sum.

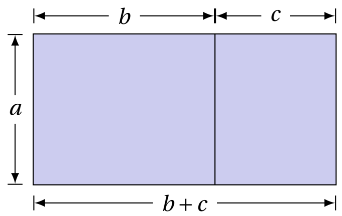

Figure D.1: The area of the entire rectangle is $a \cdot (b+c)$, but we can also find the area as the sum of the areas of the two smaller rectangles, i.e. $a \cdot b + a \cdot c$. Since it is the same area, we must have $a \cdot (b+c) = a \cdot b + a \cdot c$.

If we remember that $-x = (-1)x$, we may deduce that

$$a - (b+c) = a + (-1)(b+c) = a + (-1)b + (-1)c = a - b - c\,.$$

If a parenthesis is preceded by a "−", we can therefore remove the parenthesis if we change the sign of every term inside the parenthesis.

Example D.3
A few examples of how to remove parentheses:

$$\begin{aligned} x + (8-2x) &= x + 8 - 2x\,, \\ 8y - (y+3) &= 8y - y - 3\,, \\ 5t + (6+2t) &= 5t + 6 + 2t\,, \\ 7p - (1-6p) &= 7p - 1 + 6p\,. \end{aligned}$$

If we need to multiply a number and a sum, we also use the distributive law.

Example D.4
Some examples of multiplying a number with a sum or a difference:

$$\begin{aligned} 2(x+5) &= 2x + 2 \cdot 5 = 2x + 10\,, \\ x - 8(5+x) &= x + (-8) \cdot 5 + (-8)x = x - 40 - 8x\,, \\ y(3+y) &= 3y + y^2\,. \end{aligned}$$

A more advanced example might be:

$$\begin{aligned} 5 - ab(3b+a) &= 5 + (-ab) \cdot 3b + (-ab)a = 5 - ab \cdot 3b - aba \\ &= 5 - 3abb - aab = 5 - 3ab^2 - a^2b\,. \end{aligned}$$

It would also be of interest to know how to multiply two sums—i.e. a calculation such as $(a+b)(c+d)$. Here we use the distributive law twice to obtain

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd\,.$$

As we can see, every term in the first parenthesis is multiplied by every term in the last parenthesis. We can illustrate this in the following way:

$$(a+b)(c+d) = ac + ad + bc + bd\,.$$

All the rules involving parentheses are summed up in the following theorem:

Theorem D.5

For calculations involving parentheses we have:

1. $a+(b+c)=a+b+c$.
2. $a-(b+c)=a-b-c$.
3. $a(b+c)=ab+ac$.
4. $(a+b)(c+d)=ac+ad+bc+bd$.

Factoring

Sometimes it is a good idea to use the distributive law "backwards". Theorem D.5(3) then becomes

Theorem D.6

For the three numbers a, b and c, we have

$$ab+ac=a(b+c)\ .$$

Rewriting an expression in this way is called *factoring*,. We begin by identifying a term, which divides every term in the expression. E.g.

$$12x+18y=6\cdot 2x+6\cdot 3y=6(2x+3y)\ .$$

Here 6 divides every term in the original expression.

Example D.7
Examples of factoring could be

$$\begin{aligned}5x+15z&=5x+5\cdot 3y=5(x+3y)\ ,\\ 7a+ab&=a(7+b)\ ,\\ 3pq-5pq^2&=3pq-5pq\cdot q=pq(3-5q)\ .\end{aligned}$$

The advanced example where we factor out $2xy$, is

$$\begin{aligned}2x^2y+4xy^2-6xy&=2xxy+2\cdot 2xyy-3\cdot 2xy\\ &=2xy\cdot x+2xy\cdot 2y-2xy\cdot 3=2xy(x+2y-3)\ .\end{aligned}$$

Factoring is a useful tool in many situations. As an example we look at a fraction that can be simplified after factoring:

$$\frac{6x+9}{12}=\frac{3(2x+3)}{12}=\frac{3(2x+3)/3}{12/3}=\frac{2x+3}{4}\ .$$

D.3 QUADRATIC MULTIPLICATION FORMULAS

When we multiply two parentheses, we multiply every term in the first parenthesis with every term in the last. If some of the terms are equal, we can simplify the resulting expression.

Two examples are[5]

[5]In theses calculations we use throughout that $ab=ba$.

$$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb = a^2 + b^2 + 2ab\,,$$

and

$$(a-b)^2 = (a-b)(a-b) = aa + a(-b) + (-b)a + (-b)(-b) = a^2 + b^2 - 2ab\,.$$

If we add the terms in the first parenthesis and subtract them in the last, we get

$$(a+b)(a-b) = aa + (-b)a + ba + (-b)b = a^2 - b^2\,.$$

Collectively, these calculations yield the following theorem:

Theorem D.8

A square of a sum:

1. $a^2 + b^2 + 2ab = (a+b)^2$.

A square of a difference:

2. $a^2 + b^2 - 2ab = (a-b)^2$.

A difference of two squares:

3. $a^2 - b^2 = (a+b)(a-b)$.

Example D.9
The formulas may be used in this manner:

$$x^2 + 49 + 14x = x^2 + 7^2 + 2\cdot 7x = (x+7)^2\,,$$
$$4p^2 - 25q^2 = (2p)^2 - (5q)^2 = (2p+5q)(2p-5q)\,,$$
$$9a^2 + 36 - 36a = (3a)^2 + 6^2 - 2\cdot 3a\cdot 6 = (3a-6)^2\,.$$

Example D.10
Sometimes we can simplify fractions, even if at first it looks impossible:

$$\frac{x^2 + 25 - 10x}{4x - 20} = \frac{(x-5)^2}{4(x-5)} = \frac{x-5}{4}\,.$$

Equations

E

An equation consists of two calculations separated by an equals sign. The equals sign may be viewed as the statement that the two calculations yield the same result.

An unknown number (the *unknown*)[1] is present in at least one of the two calculations. A *solution* to the equation is a number, such that the statement is true when the number is inserted in place of the unknown.

[1] An equation might contain more than one unknown, but in the simplest case there is only one.

Example E.1
An equation could be

$$5x - 9 = 2x\,.$$

The two calculations are $5x - 9$ and $2x$. The equation states that these calculations yield the same result.

$x = 3$ is a solution to the equation, since the two sides of the equation yield

$$\begin{aligned} 5 \cdot 3 - 9 &= 6 \quad \text{(left hand side, } 5x-9\text{)}, \\ 2 \cdot 3 &= 6 \quad \text{(right hand side, } 2x\text{)}, \end{aligned}$$

when we insert 3 in place of x. I.e. the two calculations yield the same result (6), when $x = 3$.

On the other hand, $x = 7$ is not a solution, since

$$\begin{aligned} 5 \cdot 7 - 9 &= 36\,, \\ 2 \cdot 7 &= 14\,. \end{aligned}$$

Here the two sides yield different results.

E.1 SOLVING AN EQUATION

Solving an equation consists of finding those numbers that are solutions to the equation.[2] This involves a simple technique.

[2] It is possible for equations to have more than one solution; it is also possible to have equations that have no solutions.

The two sides of the equations are calcutions that yield the same result if we insert a solution in place of the unknown. E.g.

$$2 \cdot 4 + 3 = 11 \qquad \text{and} \qquad 5 \cdot 4 - 9 = 11\,,$$

which means that $x = 4$ is a solution to the equation[3]

$$2x + 3 = 5x - 9\,. \tag{E.1}$$

[3]Both sides yield 11 when we insert 4 in place of x.

But the equation

$$2x + 3 + 9 = 5x - 9 + 9$$

must have the same solution. The calculations are not the same as before, so each side no longer yields 11; but the results on either side are still equal, since we added the same number to both sides. Now when we insert $x = 4$, we get

$$2 \cdot 4 + 3 + 9 = 20 \qquad \text{and} \qquad 5 \cdot 4 - 9 + 9 = 20\,.$$

So if we add the same number to both sides of an equation, we get a new equation—but one with the same solution as the previous equation.

This reasoning also works for subtraction, multiplication, etc. We therefore have the following theorem:

Theorem E.2

If we carry out the same arithmetical operation on both sides of an equation, we get a new equation with the exact same solutions.

The equation (E.1) might be solved in the following manner:

$$\begin{aligned}
2x + 3 &= 5x - 9 && \text{Equation (E.1)}\\
2x + 3 + 9 &= 5x - 9 + 9 && \text{Add 9 to both sides.}\\
2x + 12 &= 5x && \text{Reduce}\\
2x + 12 - 2x &= 5x - 2x && \text{Subtract } 2x \text{ from both sides.}\\
12 &= 3x && \text{Reduce.}\\
\frac{12}{3} &= \frac{3x}{3} && \text{Divide by 3 on both sides.}\\
4 &= x && \text{Reduce.}
\end{aligned}$$

The last line is also an equation, but is an equation that is easy to solve. The solution to the equation $4 = x$ is $x = 4$—an this is also the solution to the original equation.[4]

[4]The point of adding, subtracting etc. is to get to an equation, which gives us the solutions directly.

We may use whichever operation we want to, but it is absolutely necessary that we always use the exact same operation on both sides of the equation.[5]

[5]It is, however, never permitted to multiply by 0, since then the equation is reduced to $0 = 0$, which is always correct—and then it is not possible to find solutions to the original equation.

When we use an operation on both sides of an equation, it is important to remember to use the operation on the *entire* side and not just a part of it. See the examples below.

Example E.3

If we want to multiply by 2 on both sides of the equation $\frac{1}{2}x + 3 = 8$, we need to use parentheses:

$$\tfrac{1}{2}x + 3 = 8 \qquad \Leftrightarrow^{6}$$

$$2 \cdot \left(\tfrac{1}{2}x + 3\right) = 2 \cdot 8 \qquad \Leftrightarrow$$
$$x + 6 = 16\,.$$

If we finish solving the equation, we find the solution $x = 10$.

[6]The sign $\Leftrightarrow$ means "if and only if". This means that the two statements on each side of the arrow are logically equal, i.e. one of the is true only if the other one is true and vice versa.

Example E.4
If we want to solve the equation $x^2 + 4 = 13$, we might be tempted first to take the square root on both sides. This yields

$$x^2 + 4 = 13 \qquad \Leftrightarrow \qquad \sqrt{x^2+4} = \sqrt{13}\,.$$

Here, we cannot reduce the left hand side, since we need to add before we take the square root—and we cannot do this, because we do not know the value of x.

It turns out that it is a better idea to first subtract 4 to get

$$x^2 + 4 = 13 \qquad \Leftrightarrow \qquad x^2 = 9\,.$$

This equation is easy to solve. Its two solutions are $x = -3$ and $x = 3$.[7]

[7]Remember that $(-3)^2 = 9$, since the product of two negative numbers is positive. Therefore $x = -3$ is also a solution to the equation.

Testing a solution

If we are given a solution to an equation, or we want to check whether a solution is correct, we may test the solution. We simply insert the solution into both sides of the equation and see if we get the same result.

Example E.5
Is $x = 2$ a solution to $x^3 - 3 = 2 \cdot x + 1$?

The left hand side yields

$$2^3 - 3 = 8 - 3 = 5\,.$$

The right hand side yields

$$2 \cdot 2 + 1 = 4 + 1 = 5\,.$$

When we insert $x = 2$, the two sides yield the same result. Therefore $x = 2$ *is* a solution to the equation.

Example E.6
Is $x = 3$ a solution to the equation $\frac{4x}{x+1} = 5$?

The left hand side yields

$$\frac{4 \cdot 3}{3+1} = \frac{12}{4} = 3\,.$$

This is not equal to 5, which is the right hand side. Therefore $x = 3$ is *not* a solution.

How about $x = -5$? Here the left hand side yields

$$\frac{4 \cdot (-5)}{-5+1} = \frac{-20}{-4} = 5\,.$$

This is equal to the right hand side, so $x = 5$ is a solution.

E.2 THE ZERO PRODUCT RULE

If we multiply by 0, the result is always 0. On the other hand, we cannot multiply two non-zero numbers and get 0 as a result. Therefore, if the result of a multiplication is 0, at least one of the numbers involved must be 0. This leads us to the following theorem:

Theorem E.7: The Zero Product Rule

If a product is 0, at least one of the factors is 0:

$$a \cdot b = 0 \qquad \Leftrightarrow \qquad a = 0 \vee b = 0\,.$$

If one side of an equation is 0, and the other side is a product, we can use this theorem to solve the equation.

Example E.8
What are the solutions to the equation $(x-3)\cdot(x+2) = 0$?

On the right hand side, we have 0, and on the left hand side the product of $x-3$ and $x+2$. According to the zero product rule, at least one of these factors must be 0, i.e.

$$x-3=0 \quad \text{or} \quad x+2=0\,,$$

which leads to the solutions

$$x=3 \quad \text{or} \quad x=-2\,.$$

Example E.9
The equation $(x+2)(x-4)(x+1) = 0$ can be solve using the zero product rule:[8]

$$\begin{aligned} (x+2)(x-4)(x+1) = 0 \qquad & \Leftrightarrow \\ x+2=0 \quad \vee \quad x-4=0 \quad \vee \quad x+1=0 \qquad & \Leftrightarrow \\ x=-2 \quad \vee \quad x=4 \quad \vee \quad x=-1\,. \end{aligned}$$

[8] The sign $\vee$, which we use below, means "or".

Sometimes we can use the zero product rule if we are able to rewrite one side of the equation as a product.

Example E.10
The equation $x^2 - 5x = 0$ can be solved in this manner:

First we factor out x

$$x\cdot(x-5) = 0\,,$$

and then we use the zero product rule

$$x=0 \quad \vee \quad x-5=0\,.$$

Therefore the equation has the two solutions

$$x=0 \quad \vee \quad x=5\,.$$

E.3 EXPONENTIAL EQUATIONS

An exonential equation is an equation, where the unknown is an exponent. E.g.

$$3^x = 7 \tag{E.2}$$

is an exponential equation. This type of equation may be solved using the function log.[9]

The equation E.2 has the solution[10]

$$x = \frac{\log(7)}{\log(3)} = 1.771\,.$$

We have the following theorem:

Theorem E.11

The exponential equation

$$a^x = c\,,$$

where a and c are positive numbers, has the solution

$$x = \frac{\log(c)}{\log(a)}\,.$$

Example E.12
The equation

$$4^x = 15$$

has the solution

$$x = \frac{\log(15)}{\log(4)} = 1.953\,.$$

Example E.13
To solve the equation $5 \cdot 6^x + 13 = 138$, we need to first get an equation of the right form:

$$\begin{aligned} 5 \cdot 6^x + 13 &= 138 \quad \Leftrightarrow \\ 5 \cdot 6^x &= 125 \quad \Leftrightarrow \\ 6^x &= 25\,. \end{aligned}$$

Now we may solve the equation using theorem E.11:

$$x = \frac{\log(25)}{\log(6)} = 1.796\,.$$

E.4 TWO EQUATIONS IN TWO UNKNOWNS

In the previous two sections, we only looked at equations in one unknown. An example of an equation in more unknowns is

$$3x - y = 4\,.$$

Here there are two unknowns, x and y. If we have one equation in two unknowns, there is an infinite amount of pairs (x, y) of solutions to the equation.

[9] "log" stands for *logarithm*. E.g. log(3) is called the *logarithm of 3*.

[10] The result of the calculation $\frac{\log(7)}{\log(3)}$ is found on a calculator.

E.g.

$$\begin{aligned} x = 5, y = 11: &\qquad 3 \cdot 5 - 11 = 4 \\ x = 1, y = -1: &\qquad 3 \cdot 1 - (-1) = 4\,. \end{aligned}$$

But if we have *two* equations in two unknowns, there is exactly one pair of numbers, which solve both equations.[11]

[11]There are a few exceptions, which are described below.

Example E.14
The two equations

$$5x - y = 3 \quad \text{and} \quad 2x + 4y = 10$$

have the solution $x = 1$ and $y = 2$ because

$$5 \cdot 1 - 2 = 3 \quad \text{and} \quad 2 \cdot 1 + 4 \cdot 2 = 10\,.$$

No other values of x and y solve both equations.

Two equations in two unknowns may also be called a *system of equations*. The system of equations in the example above has only one solution. Some systems of equations have more than one solution or no solution at all.

Example E.15
The two equations

$$x + y = 2 \quad \text{and} \quad 3x + 3y = 6\,,$$

have an infinite amount of solutions.

This is the case because the second equation is actually the same as the first multiplied by 3. Therefore the two equations have the exact same solutions, and a pair (x, y), which solves the first equation, also solves the second.

Solving a system of equations consists of finding the pair(s) of numbers, which solve(s) the system. Below we describe two methods.

The Method of Substitution

Solving two equations in two unknowns via the method of substitution is done by solving for one of the unknowns in the first equation and inserting the found expression in the second. Doing this results in an equation with just one unknown.

Example E.16
To solve the system of equations

$$2x + y = 7 \quad \text{og} \quad 5x - 3y = 12\,,$$

we solve for y in the først equation. We get

$$2x + y = 7 \quad \Leftrightarrow \quad y = 7 - 2x\,. \tag{E.3}$$

Next, we insert this expression in the second equation

$$5x - 3y = 12 \quad \Rightarrow \quad 5x - 3\big(7 - 2x\big) = 12\,.$$

We now solve this new equation:

$$\begin{aligned} 5x - 3(7 - 2x) &= 12 \\ 5x - 21 + 6x &= 12 \\ 11x - 21 &= 12 \\ 11x &= 12 + 21 \\ 11x &= 33 \\ x &= 3\,. \end{aligned}$$

From (E.3), we have $y = 7 - 2x$, i.e.

$$y = 7 - 2 \cdot 3 = 1\,.$$

Thus the solution to this system of equations is $x = 3$ and $y = 1$.

Example E.17
The system[12]

$$x + y = 5 \quad \wedge \quad y^2 = 9\,,$$

can be solve by first solving the last equation:

$$y^2 = 9 \quad \Leftrightarrow \quad y = -3 \vee y = 3\,.$$

Each of these values of y have a corresponding value of x.

The first equation may be written as $x = 5 - y$, which gives us these two values of x:

$$\begin{aligned} y &= -3 \quad \Rightarrow \quad x = 5 - (-3) = 8 \\ y &= 3 \quad\;\; \Rightarrow \quad x = 5 - 3 = 2\,. \end{aligned}$$

Therefore the system of equations has the following solution:

$$\big(x = 2 \wedge y = 3\big) \quad \vee \quad \big(x = 8 \wedge y = -3\big)\,.$$

[12] The sign $\wedge$ used below means “and”. This is an “inclusive and”, which means that the two equations on each side of $\wedge$ must be true simultaneously.

The Method of Elimination

Another method for solving two equations in two unknowns is the so-called “method of elimination”. This method only works if both equations can be written in the form

$$ax + by = c\,,$$

where a, b og c are three numbers.

The general idea is to rewrite the system of equations, such that either x or y has the same coefficient in the two equations. When we have done that, we can subtract the two equations to get a new equation with just one unknown (*eliminating* the other).

We illustrate this method through some examples

Example E.18
Here we have the system of equations

$$\begin{cases} 3x + y = 11 \\ -2x + 5y = 21 \end{cases} .$$

We now multiply the first equation by 5 on both sides. Then we get

$$\begin{cases} 15x + 5y = 55 \\ -2x + 5y = 21 \end{cases} .$$

If we subtract these two equations[13] we get the new equation

$$(15x + 5y) - (-2x + 5y) = 55 - 21 ,$$

which we can reduce to get

$$17x = 34 .$$

The solution to this equation is $x = 2$.

Now we know the value of x, so we insert this into one of the equations from the original system. Here we choose $3x + y = 11$:

$$3 \cdot 2 + y = 11 \quad \Leftrightarrow \quad y = 5 .$$

Thus the solution is $x = 2$ and $y = 5$.

In the above example, it was enough to rewrite one of the equations. Sometimes we need to rewrite them both.

Example E.19
We rewrite the system of equations

$$\begin{cases} 5x - 4y = 22 \\ -2x + 8y = 4 \end{cases}$$

by multiplying the first equation by 2 and the second by 5:[14]

$$\begin{cases} 10x - 8y = 44 \\ -10x + 40y = 20 \end{cases} .$$

The coefficients of x differ in signs, so we add the equations instead of subtracting them:

$$(10x - 8y) + (-10x + 40y) = 44 + 20 .$$

We reduce this equation and solve it:

$$32y = 64 \quad \Leftrightarrow \quad y = 2 .$$

We insert this value into one of the original equations, $5x - 4y = 22$:

$$5x - 4 \cdot 2 = 22 \quad \Leftrightarrow \quad 5x = 30 \quad \Leftrightarrow \quad x = 6 .$$

Therefore the solution to this system of equations is $x = 6$ and $y = 2$.

[13] We are allowed to subtract two equations because the left and right hand sides of an equation are actually the same number. Thus we actually subtract the same number from both sides.

[14] We multiply each equation by the coefficient of x from the other equation.

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(2) $\frac{a}{b} = \frac{k \cdot a}{k \cdot b}$

Addition (3) $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d}$

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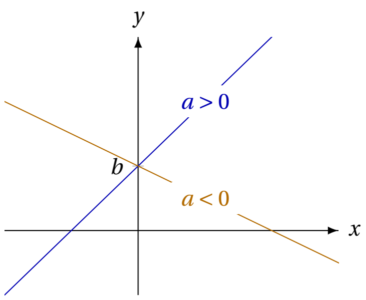

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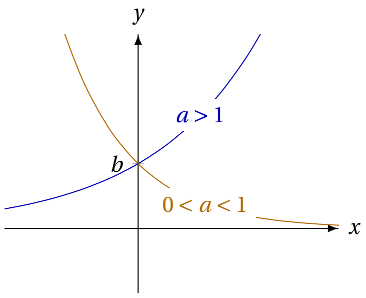
EXPONENTIAL FUNCTIONS



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