Plane Geometry

Version 1.1

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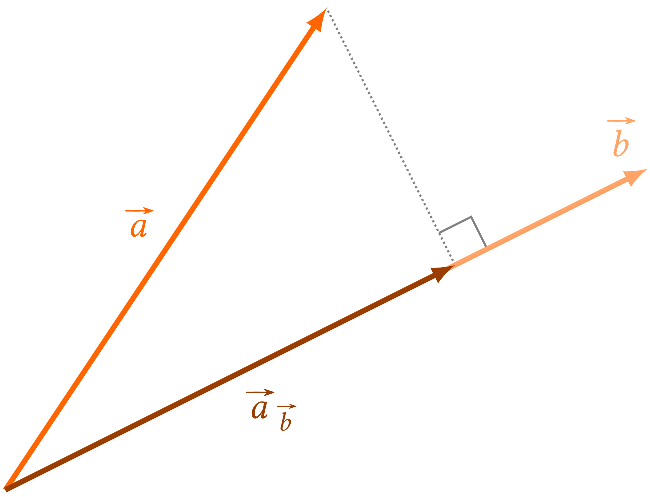

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www.mathematicus.dk

Plane Geometry
Version 1.1, 2019

These notes are a translation of the Danish "Geometri i planen" written for the Danish stx.

The notes cover plane geometry based on plane vectors; e.g. the trigonometric functions are defined without reference to the unit circle, but are instead based on angles of vectors.

The document is written using the document preparation system LaTeX, see www.tug.org and www.miktex.org. Figures and diagrams are produced using *pgf/TikZ*, see www.ctan.org/pkg/pgf.

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Contents

1 Plane vectors

When we look at a location in a 2-dimensional coordinate system, we describe the location as a point (x, y), where the x-coordinate shows the position left/right, while the y-coordinate shows the position up/down.

But if we move in a coordinate system, it is also possible to describe the movement through coordinates which show how many units we move left/right, and how many units we move up/down. We call this description a *vector*, and we often draw it as an arrow in a coordinate system. A few examples of vectors are shown in figure 1.1.

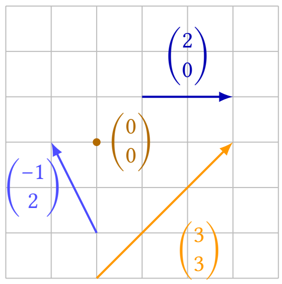


Figure 1.1: Examples of plane vectors.

Notice that the coordinates of the vector are written vertically. We write them this way so as not to confuse vectors with points. The vector

$$\begin{pmatrix}-1\\2\end{pmatrix}$$

describes the following *movement* in a coordinate system: 1 unit to the left and 2 units up. A vector is denoted as a letter with a small arrow, e.g. $\vec{a}$. The arrow above the a shows that this is a vector. So, we have the following definition of a vector:

Definition 1.1

A *vector* in the plane is a mathematical quantity, which describes movement in the plane through two coordinates:

$$\vec{a} = \begin{pmatrix}a_x\\a_y\end{pmatrix} .$$

Here, it is important to point out that a vector is only given by its coordinates, it has no location. Any arrow with the same direction and length is therefore a *representation* of the same vector. For the vectors in figure 1.2 we therefore have

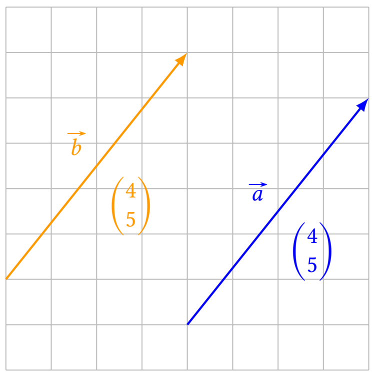


Figure 1.2: Two representations of the same vector.

$$\vec{a} = \vec{b} ,$$

because the two arrows share the same coordinates (4 to the right, 5 up). So, they represent the same vector.

For any vector we may define the so-called *opposite* vector, which is a vector of the same length pointing in the opposite direction. We define this in the following way:

Definition 1.2

If $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, we define the opposite vector of $\vec{a}$ to be

$$-\vec{a} = \begin{pmatrix} -a_x \\ -a_y \end{pmatrix} .$$

Example 1.3 If $\vec{a} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, the opposite vector is

$$-\vec{a} = \begin{pmatrix} -(-3) \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} .$$

Here we see that $\vec{a}$ describes a movement of 3 to the left and 1 up, whereas the opposite vector $-\vec{a}$ describes a movement of 3 to the right and 1 down.

Instead of describing the vectors coordinates, we may also refer to its *length* and *direction*. The length of the vector corresponds to the length of an arrow representing the vector. We may find this length using the Pythagorean theorem. If we look at figure 1.3, we see that the arrow representing the vector

$$\vec{a} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} ,$$

is the hypotenuse of a right-angled triangle with legs 3 and 4. So, we may calculate the length $|\vec{a}|$ of the vector by using the Pythagorean theorem on its coordinates:

$$|\vec{a}| = \sqrt{(-3)^2 + 4^2} = 5 .$$

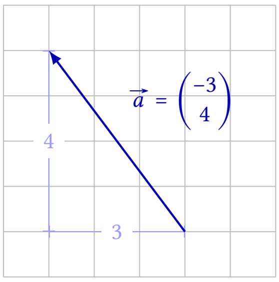


Figure 1.3: The Pythagorean theorem may be used to find the length of a vector.

Therefore, we have the following theorem:

Theorem 1.4

If the vector $\vec{a}$ has coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, its length is

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} .$$

The vector with coordinates $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a special vector. It is the only vector with length 0, and we call it the *zero vector*, $\vec{0}$:

Definition 1.5

The vector whose length is 0 is called the *zero vector*, $\vec{0}$.

This vector is also called a *trivial* vector. Since the length is 0, this vector has no direction.

If we look at two points in the plane, these two points define a vector. In figure 1.4 two representations of the vector $\overrightarrow{AB}$ are shown. The arrow from A to B determines the vector $\overrightarrow{AB}$; but since every arrow of the same direction and length are representations of the same vector, we might as well draw it somewhere else – which we have also done in the figure. We have the following definition:

Definition 1.6

If A and B are two points in the plane, the vector $\overrightarrow{AB}$ is the vector which may be represented by an arrow *from A to B*.

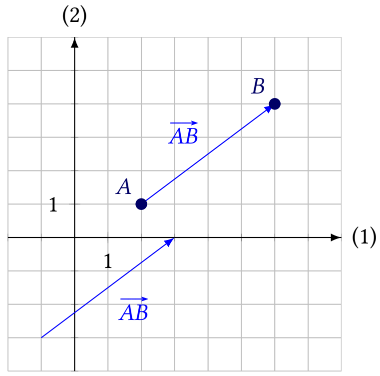


Figure 1.4: Two representations of the vector $\overrightarrow{AB}$.

The coordinates of the vector $\overrightarrow{AB}$ may be found by investigating how much the x- and y-coordinates change when we move from A to B. If the two points have coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$, the x-coordinate increases by $x_2 - x_1$ and the y-coordinate increases by $y_2 - y_1$ (see figure 1.5). Therefore, we have the following theorem:

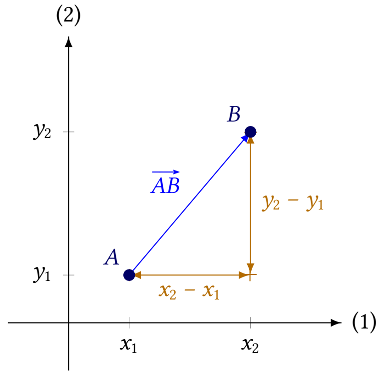


Figure 1.5: Calculating the coordinates of the vector $\overrightarrow{AB}$.

Theorem 1.7

The vector $\overrightarrow{AB}$ between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates

$$\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} .$$

If we have one point in a coordinate system, we may define a vector with the same coordinates as this point; this is a so-called *position vector*, which describes the movement from the origin (i.e. the point $(0, 0)$) to the given point (see, figure 1.6).

Definition 1.8

If $A(x_0, y_0)$ is a point in a coordinate system, we define the *position vector* of the point to be

$$\overrightarrow{OA} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} .$$

$\overrightarrow{OA}$ is the vector from $O(0, 0)$ to $A(x_0, y_0)$.

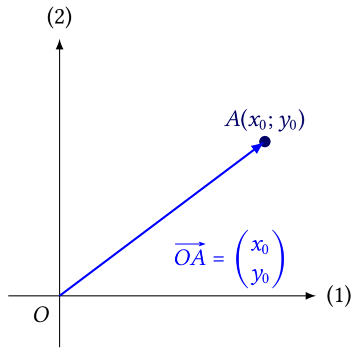


Figure 1.6: The vector $\overrightarrow{OA}$ has the same coordinates as the point A.

1.1 Addition and subtraction

In this section, we describe how to calculate with vectors. Vectors may be added and subtracted. These operations are defined simply by adding or subtracting the separate coordinates.

Definition 1.9

Given two vectors

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix},$$

we define

$$\vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix} \qquad \text{and} \qquad \vec{a} - \vec{b} = \begin{pmatrix} a_x - b_x \\ a_y - b_y \end{pmatrix}.$$

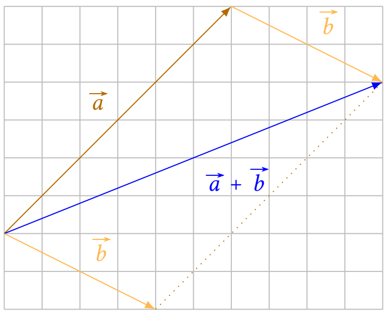


Figure 1.7: Addition of vectors.

By analysing figure 1.7 we arrive relatively easy at this theorem:

Theorem 1.10

The sum $\vec{a} + \vec{b}$ of the two vectors $\vec{a}$ and $\vec{b}$ is the vector we get by drawing an arrow from the tail end of $\vec{a}$ to the head of $\vec{b}$, when vector $\vec{a}$ and $\vec{b}$ are drawn head to tail.

We may also find $\vec{a} + \vec{b}$ as the diagonal of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$ when they are drawn tail to tail. (We sometimes call this the "parallelogram law".)

Subtraction of vectors may also be interpreted geometrically. If we look at the calculation

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}),$$

we see that we subtract a vector from another by adding the opposite vector. I.e. we have the following theorem (see also figure 1.8):

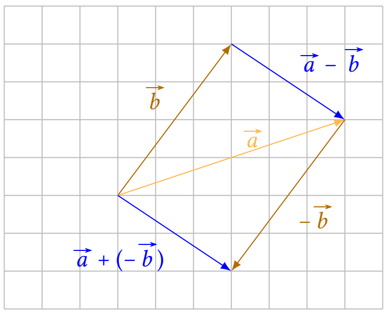


Figure 1.8: Subtraction of vectors.

Theorem 1.11

The difference $\vec{a} - \vec{b}$ between the two vectors $\vec{a}$ and $\vec{b}$ is the vector we get by drawing an arrow from the head of $\vec{b}$ to the head of $\vec{a}$, when the two vectors are drawn tail to tail.

We may also multiply a vector by a number. We define this in the following way:

Definition 1.12

Let $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ be a vector and let t be a number. Then

$$t \cdot \vec{a} = \begin{pmatrix} t \cdot a_x \\ t \cdot a_y \end{pmatrix}.$$

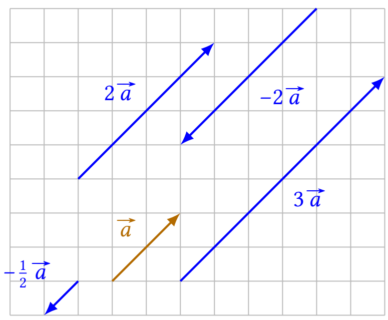


Figure 1.9: Multiplication of a vector by a number.

From this definition, it follows that multiplying a vector by t yields a new vector which is t times the length of the original—and if $t < 0$, the vector changes direction. This is illustrated in figure 1.9.

For vectors between points we have the following important theorem, which is illustrated in figure 1.10:

Theorem 1.13

If A, B and C are three points in the plane, we have

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}\,.$$

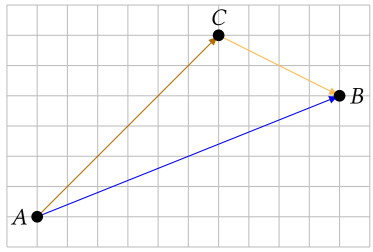


Figure 1.10: $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$.

Proof
Let the three points have coordinates $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Then

$$\begin{aligned}\overrightarrow{AC} + \overrightarrow{CB} &= \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix} + \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \\ &= \begin{pmatrix} x_2 - x_3 + x_3 - x_1 \\ y_2 - y_3 + y_3 - y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \overrightarrow{AB}\,.\end{aligned}$$

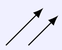
Since we have $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$, we have proven the theorem. ■

1.2 Vectors and Angles

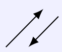
Eventhough vectors have no specific location but instead describe movement, it is still possible to compare them geometrically. Since vectors describe direction, we may look at the angle they form, and whether they are parallel or orthogonal. The following definition describes the different cases:

Definition 1.14

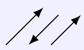
We say that two vectors $\vec{a}$ are $\vec{b}$



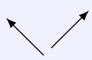
in the same direction
if $\vec{a}$ and $\vec{b}$ point in the same parallel direction.



in opposite directions
if $\vec{a}$ and $\vec{b}$ point in opposite parallel directions.



parallel, $\vec{a} \parallel \vec{b}$
if $\vec{a}$ and $\vec{b}$ are in the same or the opposite direction.



orthogonal, $\vec{a} \perp \vec{b}$
if $\vec{a}$ is perpendicular to $\vec{b}$.

Because the zero vector has no direction, this vector is neither parallel nor perpendicular to any other vector.

The definition above deals implicitly with angles between vectors. If we want to measure the angle between two vectors directly, we can do this in different ways. We can either measure the smallest possible angle, or we can measure the angle by moving in a certain direction from one vector to the other.

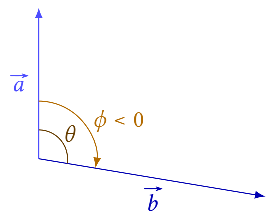


Figure 1.11: The angle θ is the angle between $\vec{a}$ and $\vec{b}$, and ϕ is the angle from $\vec{a}$ to $\vec{b}$.

Definition 1.15: Angles between vectors

If $\vec{a}$ and $\vec{b}$ are two vectors, we define

1. the angle *between* the vectors $\vec{a}$ and $\vec{b}$ to be the smallest angle spanned by the vectors, and
2. the angle *from* $\vec{a}$ *to* $\vec{b}$ to be the signed angle, we get by moving from vector $\vec{a}$ to vector $\vec{b}$.[1]

The angle between the vectors $\vec{a}$ and $\vec{b}$ is also denoted by $\angle(\vec{a}, \vec{b})$. The difference between these two angles is shown in figure 1.11.

[1]The positive direction of rotation is always *counter-clockwise*, so if we move clockwise the angle is negative.

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is an angle between $0°$ and $180°$, while the angle from $\vec{a}$ to $\vec{b}$ is an angle between $-180°$ and $180°$.

In the next chapter, we describe the relations between coordinates and angles in greater detail.

1.3 Exercises

Exercise 1.1
Determine the opposite vector of the following vectors:

a) $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ b) $\vec{b} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$

c) $\vec{c} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ d) $\vec{d} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$

Exercise 1.2
Determine the length of the vectors in exercise 1.1.

Exercise 1.3
The points A, B, C and D have coordinates $A(2, 1)$, $B(3, 5)$, $C(-4, 0)$ and $D(2, 9)$. Determine the coordinates of the following vectors:

a) $\overrightarrow{AB}$ b) $\overrightarrow{BC}$

c) $\overrightarrow{DC}$ d) $\overrightarrow{AD}$

e) $\overrightarrow{CA}$ f) $\overrightarrow{BD}$

Exercise 1.4
Given the three vectors

$$\vec{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} 0 \\ 5 \end{pmatrix},$$

calculate the vectors

a) $\vec{a} + \vec{b}$ b) $2 \cdot \vec{a}$

c) $\vec{b} - \vec{c}$ d) $\vec{c} + 3 \cdot \vec{a}$

e) $\vec{a} - \vec{b} + \vec{c}$ f) $4 \cdot \vec{a} + 3 \cdot \vec{c} - 5 \cdot \vec{b}$

Exercise 1.5
Reduce as much as possible:

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

b) $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{BB} + \overrightarrow{DA}$

c) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{DC}$

2 Angles in the plane

In this chapter, we look at the connection between vector coordinates and angles. In order to describe this connection, we first need to introduce so-called *unit vectors*.

2.1 Unit Vectors

When we look at angles in the plane, we only need to look at the direction of the vectors. At first, we therefore look only at *unit vectors*, which are vectors of length 1:

Definition 2.1

A *unit vector* is a vector $\vec{e}$ of length 1.

If $\vec{a}$ is a vector, the vector $\vec{e}_{\vec{a}}$ is a unit vector in the direction of $\vec{a}$, i.e.

$$\vec{e}_{\vec{a}} = \frac{1}{|\vec{a}|}\,\vec{a}\ .$$

In the usual coordinate system in the plane, two directions are special: Those given by the x- and the y-axis. We therefore define two special unit vectors in these directions:

Definition 2.2

In the usual coordinate system in the plane $\vec{e}_x$ is a unit vector in the direction of the x-axis, and $\vec{e}_y$ is a unit vector in the direction of the y-axis, i.e.

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ .$$

Since every unit vector has length 1, the only difference between unit vectors is there direction. This direction may be described by an angle. We therefore define the *directional angle* og a vector as follows:

Definition 2.3

The *directional angle* of a vector $\vec{a}$ is the angle from $\vec{e}_x$ to $\vec{a}$.

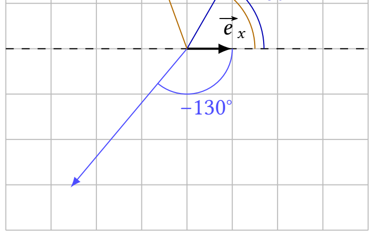


Figure 2.1: The vector $\vec{e}_x$ and the directional angles of different vectors.

Figure 2.1 shows three different vectors and their directional angles. Remember that counter-clockwise angles are positive, and clockwise angles are negative.

2.2 Cosine, Sine and Tangent

Unit vectors and directional angles may be used to define the so-called *trigonometric* functions *cosine* and *sine*. The two functions are used to calculate x- and y-coordinates of a unit vector from the corresponding directional angle.

Definition 2.4

Let the unit vector $\vec{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$ have directional angle θ. The three trigonometric functions cos, sin and tan are then defined to be

1. $\cos(\theta) = e_x$,
2. $\sin(\theta) = e_y$, og
3. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

Note that $\tan(\theta)$ is undefined when $\cos(\theta) = 0$, i.e. if $\theta = \pm 90°$.

The three functions are built in to most calculators and every CAS.

Note that because $\cos(\theta)$ and $\sin(\theta)$ are x- and y-coordinates of a unit vector, $\cos(\theta)$ and $\sin(\theta)$ may only assume values between −1 and 1, i.e.

$$-1 \leq \cos(v) \leq 1 \qquad \text{and} \qquad -1 \leq \sin(v) \leq 1 \,.$$

Example 2.5 The coordinates of the unit vector $\vec{e}_{130°}$ with directional angle $\theta = 130°$ are

$$\vec{e}_{130°} = \begin{pmatrix} \cos(130°) \\ \sin(130°) \end{pmatrix} = \begin{pmatrix} -0.643 \\ 0.766 \end{pmatrix} .$$

This vector and two other unit vectors are shown in figure 2.2.

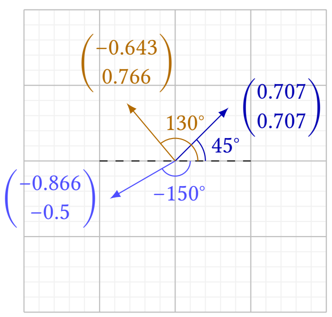


Figure 2.2: Some unit vectors and their coordinates.

Analysing symmetri in the plane allows us to prove the following theorem:

Theorem 2.6

The following hold:

1. $\cos(-\theta) = \cos(\theta)$
2. $\sin(-\theta) = -\sin(\theta)$
3. $\cos(180° - \theta) = -\cos(\theta)$
4. $\sin(180° - \theta) = \sin(\theta)$
5. $\cos(90° - \theta) = \sin(\theta)$
6. $\sin(90° - \theta) = \cos(\theta)$

Since the length of a unit vector is 1, we also have the following theorem:

Theorem 2.7: Pythagorean identity

The following holds:

$$\cos(v)^2 + \sin(v)^2 = 1 \ .$$

Any vector may be described through its length and its directional angle, and the functions cos and sin may be used to translate this into coordinates.

Example 2.8 The vector $\vec{a}$ has length $|\vec{a}| = 4$ and directional angle $\theta = 35°$. We may then find the coordinates like this:[1]

$$\vec{a} = |\vec{a}|\,\vec{e}_{\vec{a}} = 4 \cdot \begin{pmatrix} \cos(35°) \\ \sin(35°) \end{pmatrix} = \begin{pmatrix} 4 \cdot 0.819 \\ 4 \cdot 0.574 \end{pmatrix} = \begin{pmatrix} 3.28 \\ 2.29 \end{pmatrix} \ .$$

[1]Remember that $\vec{e}_{\vec{a}}$ unit vector in the direction of $\vec{a}$ (see definition 2.4).

2.3 Inverse Trigonometric Functions

In example 2.8, we calculated a vectors coordinates from its directional angle and length. It would be convenient to be able to translate the other way—i.e. to find the length and the directional angle based on the coordinates.

The length of a vector may be calculated from the formula in theorem 1.4, but what about the angle? If we have a unit vector $\vec{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$, then

$$\begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \ ,$$

i.e.

$$\cos(\theta) = e_x \qquad \text{and} \qquad \sin(\theta) = e_y \ ,$$

To find the directional angle θ, we need to be able to solve such equations. So, we need functions which do the opposite of cos and sin.

The following example shows that it is non-trivial to construct such functions:

Example 2.9 Let $\vec{e}_1$ be the unit[2] vector with directional angle 20° and $\vec{e}_2$ be the unit vector with directional angle −20°. Then the two vectors have coordinates

$$\vec{e}_1 = \begin{pmatrix} \cos(20°) \\ \sin(20°) \end{pmatrix} = \begin{pmatrix} 0.940 \\ 0.342 \end{pmatrix}$$
$$\vec{e}_2 = \begin{pmatrix} \cos(-20°) \\ \sin(-20°) \end{pmatrix} = \begin{pmatrix} 0.940 \\ -0.342 \end{pmatrix} \ .$$

The two vectors $\vec{e}_1$ and $\vec{e}_2$ have different directional angles, but the same x-coordinates.

If $\vec{e}_3$ and $\vec{e}_4$ are unit vectors with directional angles of 50° and 130°, their coordinates are

$$\vec{e}_3 = \begin{pmatrix} \cos(50°) \\ \sin(50°) \end{pmatrix} = \begin{pmatrix} 0.643 \\ 0.766 \end{pmatrix}$$

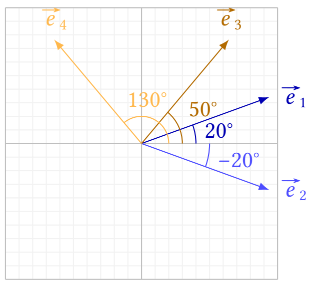


Figure 2.3: Unit vectors with the same x- or y-coordinates.

$$\vec{e}_4 = \begin{pmatrix} \cos(130^\circ) \\ \sin(130^\circ) \end{pmatrix} = \begin{pmatrix} -0.643 \\ 0.766 \end{pmatrix} .$$

Here, we have unit vectors with different directional angles, but the same y-coordinates.

The four vectors are shown in figure 2.3.

The so-called *inverse* trigonometric functions are therefore only able to give one of two possible directional angles when we know the x- or y-coordinate. The functions are defined in the following way:[2]

[2]The two functions are sometimes also denoted by arccos and arcsin, because they give the angle (the "arc") when the cosine or the sine is known.

Computer programmes/CAS's usually denote the two functions by asin and acos.

Definition 2.10

The function $\cos^{-1}$ is called the *inverse cosine.* $\cos^{-1}(t)$ is the angle θ in the interval between 0° and 180° which solves the equation $\cos(\theta) = t$.

The function sin -1 is called the *inverse sine.* $\sin^{-1}(t)$ is the angle in the interval between -90° and 90° which solves the equation $\sin(\theta) = t$.

Example 2.11 If we want to solve the equation

$$\cos(\theta) = 0.5 ,$$

we use a calculator and find that

$$\theta = \cos^{-1}(0.5) = 60^\circ .$$

Thus, $\theta = 60^\circ$ is a solution to this equation.

If we solve equations with cosines or sines in this way, we need to realise that we do not find every solution.

Example 2.9 shows that -60° is also a solution to the equation $\cos(\theta) = 0.5$. Therefore the equation should have been solved like this:

$$\cos(\theta) = 0.5 \qquad \Longleftrightarrow \qquad \theta = \pm\cos^{-1}(0.5) \qquad \Longleftrightarrow \qquad \theta = \pm 60^\circ .$$

(The sign $\pm$ means that there are two solutions: one positive, and one negative.)

Now, we may derive a formula for the directional angle of a vector based on its coordinates:

Theorem 2.12

Let the vector $\vec{a}$ be given by the coordinates

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} .$$

Then, the directional angle of the vector is

$$\theta = \begin{cases} \cos^{-1}\left(\frac{a_x}{|\vec{a}|}\right) & \text{if } a_y \geq 0 \\ -\cos^{-1}\left(\frac{a_x}{|\vec{a}|}\right) & \text{if } a_y < 0 \end{cases} .$$

Proof
If $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, a unit vector in the direction of $\vec{a}$ is given by

$$\vec{e}_{\vec{a}} = \frac{1}{|\vec{a}|} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} \frac{a_x}{|\vec{a}|} \\ \frac{a_y}{|\vec{a}|} \end{pmatrix} .$$

The unit vector also has coordinates $\vec{e}_{\vec{a}} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$, where θ is the directional angle of $\vec{a}$. I.e.

$$\cos(\theta) = \frac{a_x}{|\vec{a}|} \qquad \Longleftrightarrow \qquad \theta = \pm \cos^{-1}\left(\frac{a_x}{|\vec{a}|} \right) .$$

If the vector $\vec{a}$ has a positive y-coordinate ($a_y \geq 0$), the directional angle is positive, i.e. the solution with + is correct. The other solution is correct if $a_y < 0$. This proves the theorem. ■

Example 2.13 In this example, we determine the length and the directional angle of the two vectors

$$\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 2 \\ -7 \end{pmatrix} .$$

The length of vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{3^2 + 4^2} = 5 ,$$

and the directional angle is

$$\theta = \cos^{-1}\left(\frac{3}{5} \right) = 53,13^\circ .$$

The length of vector $\vec{b}$ is

$$\left|\vec{b}\right| = \sqrt{2^2 + (-7)^2} = \sqrt{53} = 7,28 ,$$

and the directional angle is

$$\phi = -\cos^{-1}\left(\frac{2}{7,28} \right) = -74,05^\circ .$$

Note that the directional angle of $\vec{b}$ is negative because vector $\vec{b}$ has a negative y-coordinate.

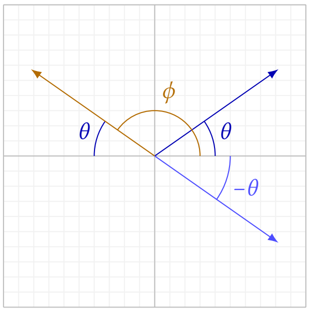
2.4 Exercises

Exercise 2.1
Determine the coordinates of the unit vector with directional angle

a) 15°
b) 60°
c) -30°
d) 145°

and draw the vectors in a coordinate system.

Exercise 2.2
The figure below shows three unit vectors with directional angles θ, $-\theta$ and $\phi = 180^\circ - \theta$.



a) Use the figure to provide an argument for theorem 2.6.

Exercise 2.3
Determine the coordinates of the vector $\vec{a}$ with directional angle θ when

a) $|\vec{a}| = 4$ and $\theta = 36^\circ$
b) $|\vec{a}| = 5,3$ and $\theta = 100^\circ$
c) $|\vec{a}| = 12$ and $\theta = -120^\circ$

Exercise 2.4
Three vectors are given:

$$\vec{a} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}.$$

Draw the following vectors in a coordinate system and calculate their lengths and directional angles:

a) $\vec{a}$
b) $\vec{b}$
c) $\vec{c}$
d) $\vec{a} + \vec{b}$
e) $\vec{b} - \vec{c}$
f) $2 \cdot \vec{a} - 5 \cdot \vec{b}$
g) $\vec{a} + \vec{b} + 2\vec{c}$

3 Scalar Product and Determinant

Until now, we have only looked at addition and subtraction of vectors, but there are other operations. These are the so-called scalar product and the determinant, which we examine in this chapter.

3.1 Scalar Product

We cannot multiply two vectors and get a new vector, but there is a form of multiplication called the *scalar product*,[1] which yields a scalar, i.e. a number, as a result.

[1] It is also sometimes called the "dot product", because the symbol is a dot.

Definition 3.1: Scalar product

If the two vectors $\vec{a}$ and $\vec{b}$ are given by the coordinates

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} ,$$

the *scalar product* of the two vectors is the number

$$\vec{a} \bullet \vec{b} = a_x b_x + a_y b_y \ .$$

Example 3.2 If

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 8 \\ -5 \end{pmatrix} ,$$

then

$$\vec{a} \bullet \vec{b} = 3 \cdot 8 + 2 \cdot (-5) = 24 + (-10) = 14 \ .$$

Note that when we multiply ordinary variables (which represent numbers), we usually omit the multiplication sign and write e.g. ab instead of $a \cdot b$; but when we calculate the scalar product, we *must* write the multiplication symbol.

The following rules hold for the scalar product:

Theorem 3.3

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors, then

1. $\vec{a} \bullet \vec{a} = |\vec{a}|^2$. (length and scalar product)
2. $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$. (the *commutative* law)
3. $\vec{a} \bullet \left(\vec{b} + \vec{c}\right) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$. } (the *distributive* law)
4. $\left(\vec{a} + \vec{b}\right) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. }
5. $\left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a} + \vec{b}\right) = |\vec{a}|^2 + \left|\vec{b}\right|^2 + 2 \cdot \vec{a} \bullet \vec{b}$.
6. $\left(\vec{a} - \vec{b}\right) \bullet \left(\vec{a} - \vec{b}\right) = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2 \cdot \vec{a} \bullet \vec{b}$.
7. $\left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a} - \vec{b}\right) = |\vec{a}|^2 - \left|\vec{b}\right|^2$.

Proof

All of the rules may be proven by coordinate calculations. Here, we prove 1, 3 and 5; the rest are left as en exercise for the reader.

If $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, then

$$\vec{a} \bullet \vec{a} = a_x a_x + a_y a_y = a_x^2 + a_y^2 = \left(\sqrt{a_x^2 + a_y^2}\right)^2 = |\vec{a}|^2 \ .$$

This proves 1.

If the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are given by the coordinates

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} , \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} ,$$

then

$$\begin{aligned} \vec{a} \bullet \left(\vec{b} + \vec{c}\right) &= \begin{pmatrix} a_x \\ a_y \end{pmatrix} \bullet \begin{pmatrix} b_x + c_x \\ b_y + c_y \end{pmatrix} \\ &= a_x(b_x + c_x) + a_y(b_y + c_y) \\ &= a_x b_x + a_x c_x + a_y b_y + a_y c_y \\ &= a_x b_x + a_y b_y + a_x c_x + a_y c_y \\ &= \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c} , \end{aligned}$$

which proves 3.

Given the two vectors $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ og $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, we have

$$\begin{aligned} \left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a} + \vec{b}\right) &= \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix} \bullet \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix} \\ &= (a_x + b_x)(a_x + b_x) + (a_y + b_y)(a_y + b_y) \\ &= a_x^2 + b_x^2 + 2a_x b_x + a_y^2 + b_y^2 + 2a_y b_y \\ &= (a_x^2 + a_y^2) + (b_x^2 + b_y^2) + 2(a_x b_x + a_y b_y) \end{aligned}$$

$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 + 2 \cdot \vec{a} \bullet \vec{b} \,,$$

and this proves 5. ■

The scalar product has some useful properties. The number gives us some information about the placement of the two vectors. In figure 3.1, we see two vectors $\vec{a}$ and $\vec{b}$, where $\vec{a}$ is parallel to $\vec{e}_x$, and the vector $\vec{b}$ has directional angle θ. We therefore know the coordinates of these two vectors to be

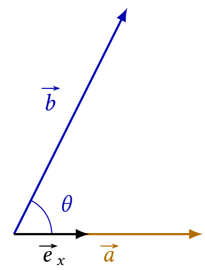


Figure 3.1: The vectors $\vec{a}$ and $\vec{b}$.

$$\vec{a} = \begin{pmatrix} |\vec{a}| \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} \left|\vec{b}\right| \cdot \cos(\theta) \\ \left|\vec{b}\right| \cdot \sin(\theta) \end{pmatrix} .$$

The scalar product of these two vectors is then

$$\vec{a} \bullet \vec{b} = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos(\theta) + 0 \cdot \left|\vec{b}\right| \cdot \sin(\theta) = |\vec{a}|\left|\vec{b}\right| \cos(\theta) \,,$$

where θ is the angle between the two vectors.

It turns out that this formula is also correct, even if $\vec{a}$ does not have the same direction as $\vec{e}_x$. We have the following theorem:

Theorem 3.4

If $\theta = \angle(\vec{a}, \vec{b})$, then

$$\vec{a} \bullet \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos(\theta) \,.$$

If we rewrite the formula in this theorem, we get

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|\left|\vec{b}\right|} \,,$$

which may be used directly to determine the angle between two vectors.

It is worth mentioning here that theorem 3.4 is also true if we use the signed angle, since $\cos(-\theta) = \cos(\theta)$.

Example 3.5 Here we find the angle between the two vectors

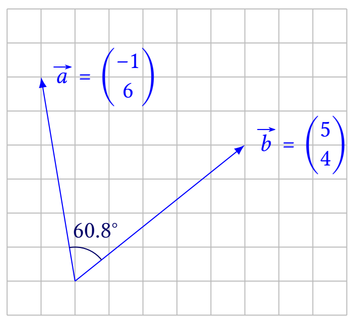


Figure 3.2: The angle between the vectors $\vec{a}$ and $\vec{b}$ is 60.8°.

$$\vec{a} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} .$$

First, we find the lengths of the two vectors

$$\begin{aligned} |\vec{a}| &= \sqrt{(-1)^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37} \\ \left|\vec{b}\right| &= \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41} \,. \end{aligned}$$

We insert these into the formula:

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|\left|\vec{b}\right|} = \frac{\begin{pmatrix} -1 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \end{pmatrix}}{\sqrt{37}\sqrt{41}} = \frac{-1 \cdot 5 + 6 \cdot 4}{\sqrt{37 \cdot 41}} = \frac{19}{\sqrt{1517}} \,,$$

and find the angle θ

$$\theta = \cos^{-1}\left(\frac{19}{\sqrt{1517}}\right) = 60.8^\circ .$$

Therefore, the angle between the two vectors is 60.8° (see figure 3.2).

The angle between two vectors is between 0° and 180°. If, for some angle θ, $0^\circ \leq \theta < 90^\circ$, $\cos(\theta) > 0$. If $90^\circ < \theta \leq 180^\circ$, then $\cos(\theta) < 0$. In the special case $\theta = 90^\circ$, we have $\cos(\theta) = 0$. Therefore, theorem 3.4 leads to the following:

Theorem 3.6

Let θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$. Then we have:

1. If $\vec{a} \cdot \vec{b} > 0$, then $0^\circ \leq \theta < 90^\circ$.
2. If $\vec{a} \cdot \vec{b} = 0$, then $\theta = 90^\circ$, i.e. $\vec{a} \perp \vec{b}$.
3. If $\vec{a} \cdot \vec{b} < 0$, then $90^\circ < \theta \leq 180^\circ$.

This theorem provides an easy test for orthogonality of two vectors. We need only calculate their scalar product. If this is 0, the vectors are orthogonal, otherwise they are not.

Example 3.7 For the two vectors $\vec{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 2 \cdot 3 + 6 \cdot (-1) = 0 .$$

Since $\vec{a} \cdot \vec{b} = 0$, these vectors are orthogonal.

Example 3.8 Given the two vectors

$$\vec{a} = \begin{pmatrix} 2 \\ t \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} ,$$

where t is a number. If the two vectors are orthogonal, what is the value of t?

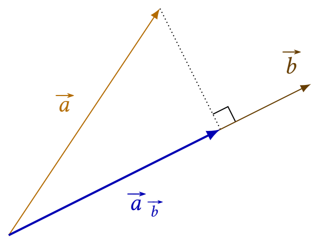
First, we calculate

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 2 \\ t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2 \cdot 3 + t \cdot 4 = 6 + 4t .$$

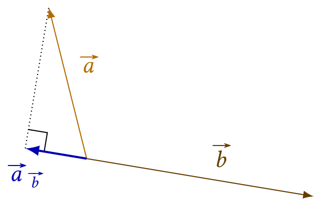
The two vectors are orthogonal, so $\vec{a} \cdot \vec{b} = 0$, i.e.

$$6 + 4t = 0 \quad \Longleftrightarrow \quad 4t = -6 \quad \Longleftrightarrow \quad t = -\frac{6}{4} = -\frac{3}{2} .$$

If the two vectors are orthogonal, the value of t must be $t = -\frac{3}{2}$.



(a) Acute angle.



(b) Obtuse angle.

Figure 3.3: The projection of $\vec{a}$ onto $\vec{b}$ when the angle between them is acute resp. obtuse.

3.2 Vector Projection

The orthogonal *projection* of one vector $\vec{a}$ onto another vector $\vec{b}$, is the vector we get, when we plac the vectors tail to tail and "collapse" $\vec{a}$ orthogonally onto $\vec{b}$. In figure 3.3, we see a representation of vector $\vec{a}_{\vec{b}}$, the projection of $\vec{a}$ onto $\vec{b}$, when the angle between the two vectors is acute, and when it is obtuse.

As the figure shows, $\vec{a}_{\vec{b}} \parallel \vec{b}$, and the two vectors are in the same direction if the angle between $\vec{a}$ and $\vec{b}$ is acute, and in opposite directions if the angle is obtuse. The coordinates of $\vec{a}_{\vec{b}}$ are given by the following theorem:

Theorem 3.9

For the projection $\vec{a}_{\vec{b}}$ of vector $\vec{a}$ onto $\vec{b}$, we have

$$\vec{a}_{\vec{b}} = \frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2} \cdot \vec{b} \ ,$$

and

$$\left|\vec{a}_{\vec{b}}\right| = \frac{\left|\vec{a} \bullet \vec{b}\right|}{\left|\vec{b}\right|} \ .$$

Proof

Figure 3.4 shows the obtuse case. A vector $\vec{c}$ is also drawn perpendicular to $\vec{b}$, such that

$$\vec{a}_{\vec{b}} + \vec{c} = \vec{a} \qquad \Longleftrightarrow \qquad \vec{c} = \vec{a} - \vec{a}_{\vec{b}} \ .$$

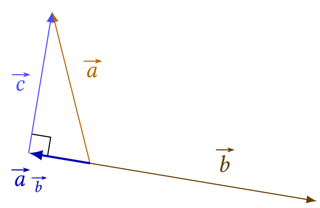


Figure 3.4: The projection of $\vec{a}$ onto $\vec{b}$ when the angle between $\vec{a}$ and $\vec{b}$ is obtuse.

Since $\vec{a}_{\vec{b}} \parallel \vec{b}$, a number t exists where

$$\vec{a}_{\vec{b}} = t\,\vec{b} \ ,$$

i.e.

$$\vec{c} = \vec{a} - t\,\vec{b} \ .$$

If we take the scalar product with $\vec{b}$ on both sides of this equation

$$\vec{c} \bullet \vec{b} = (\vec{a} - t\,\vec{b}) \bullet \vec{b} \ .$$

But $\vec{c} \bullet \vec{b} = 0$ because these two vectors are orthogonal, so we obtain the equation

$$\begin{aligned} 0 &= (\vec{a} - t\,\vec{b}) \bullet \vec{b} && \Longleftrightarrow \\ 0 &= \vec{a} \bullet \vec{b} - t\,\vec{b} \bullet \vec{b} && \Longleftrightarrow \\ 0 &= \vec{a} \bullet \vec{b} - t\left|\vec{b}\right|^2 && \Longleftrightarrow \\ t &= \frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2} . \end{aligned}$$

Since $\vec{a}_{\vec{b}} = t\,\vec{b}$, we get

$$\vec{a}_{\vec{b}} = \frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2} \cdot \vec{b} .$$

The second part of the theorem concerns the length of the projection vector. We prove this formula by calculating the length on both sides of our equation for the projection vector. We get

$$|\vec{a}_{\vec{b}}| = \left| \frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2} \cdot \vec{b} \right| = \frac{\left|\vec{a} \bullet \vec{b}\right|}{\left|\vec{b}\right|^2} \left|\vec{b}\right| = \frac{\left|\vec{a} \bullet \vec{b}\right|}{\left|\vec{b}\right|} ,$$

which proves the theorem. ∎

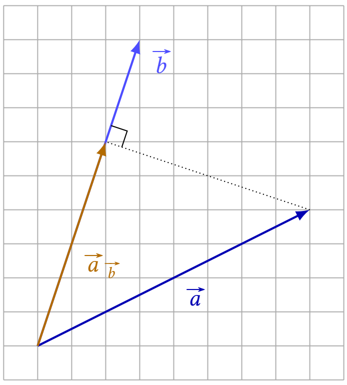


Figure 3.5: The projection of $\vec{a}$ onto $\vec{b}$.

Example 3.10 If

$$\vec{a} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} ,$$

then the projection of $\vec{a}$ onto $\vec{b}$ is

$$\vec{a}_{\vec{b}} = \frac{\begin{pmatrix} 8 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 9 \end{pmatrix}}{\sqrt{3^2 + 9^2}^2} \cdot \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{8 \cdot 3 + 4 \cdot 9}{9 + 81} \cdot \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{60}{90} \cdot \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} .$$

The vectors are shown in figure 3.5.

3.3 Determinant

The scalar product lets us determine whether two vectors are orthogonal. Another quantity, the *determinant*, allows us to determine whether they are parallel. However, before we define this quantity, we need the following definition:

Definition 3.11

For a vector $\vec{a}$, we define the *perpendicular vector* $\widehat{\vec{a}}$ to be the vector obtained by rotating $\vec{a}$ 90° in the positive direction (i.e. counter-clockwise).

An example of a perpendicular vector is shown in figure 3.6.

The coordinates of a perpendicular vector may be calculated from the coordinates of the original vector:

Theorem 3.12

If $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, then $\widehat{\vec{a}} = \begin{pmatrix} -a_y \\ a_x \end{pmatrix}$.

Proof (sketch)

Figure 3.7 shows the two vectors $\vec{a}$ and $\widehat{\vec{a}}$ drawn in a coordinate system. The figure suggests that

$$\widehat{\vec{a}} = \begin{pmatrix} -a_y \\ a_x \end{pmatrix} . \qquad \blacksquare$$

The following theorem, which may be proven by coordinate calculations, applies to perpendicular vectors:

Theorem 3.13

Given two vectors $\vec{a}$ and $\vec{b}$, we have

1. $\widehat{\widehat{\vec{a}}} = -\vec{a}$, and
2. $\widehat{\vec{a} + \vec{b}} = \widehat{\vec{a}} + \widehat{\vec{b}}$.

We now use the perpendicular vector and the scalar product to define the determinant of two vectors $\vec{a}$ and $\vec{b}$. We have

Definition 3.14

Given two vectors $\vec{a}$ and $\vec{b}$, we define their *determinant*,

$$\det\left(\vec{a}, \vec{b}\right) = \widehat{\vec{a}} \cdot \vec{b} .$$

If the coordinates of the two vectors are $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, we also denote the determinant by

$$\det\left(\vec{a}, \vec{b}\right) = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} = a_x b_y - a_y b_x .$$

The notation $\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}$ simplifies the calculation of the determinant. The "box" is calculated by first finding the product of the diagonal from the top left to the bottom right corner, and then subtracting the product of the diangonal from the bottom left corner to the top right:

$$\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} = a_x b_y - a_y b_x .$$

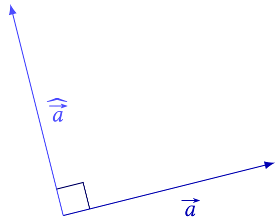


Figure 3.6: The perpendicular vector $\widehat{\vec{a}}$ of $\vec{a}$.

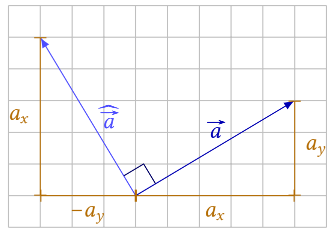


Figure 3.7: The coordinates of $\widehat{\vec{a}}$ may be found from the coordinates of $\vec{a}$.

Example 3.15 The determinant of the two vectors

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

is

$$\det\left(\vec{a}, \vec{b}\right) = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 \cdot 1 - 2 \cdot (-4) = 3 - (-8) = 11 \; .$$

When we calculate the determinant of two vectors $\vec{a}$ and $\vec{b}$, the order matters. The following theorem lists a number of rules for the determinant:

Theorem 3.16

Given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, we have

1. $\det\left(\vec{b}, \vec{a}\right) = -\det\left(\vec{a}, \vec{b}\right)$,
2. $\det\left(\vec{a} + \vec{b}, \vec{c}\right) = \det\left(\vec{a}, \vec{c}\right) + \det\left(\vec{b}, \vec{c}\right)$,
3. $\det\left(\vec{a}, \vec{b} + \vec{c}\right) = \det\left(\vec{a}, \vec{b}\right) + \det\left(\vec{a}, \vec{c}\right)$.

Proof
The first part may be proven by coordinate calculations. If

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} \; ,$$

then

$$\begin{aligned} \det\left(\vec{b}, \vec{a}\right) &= \begin{vmatrix} b_x & a_x \\ b_y & a_y \end{vmatrix} = b_x a_y - b_y a_x \\ &= a_y b_x - a_x b_y = -(a_x b_y - a_y b_x) = -\det\left(\vec{a}, \vec{b}\right) \; . \end{aligned}$$

For the determinant $\det\left(\vec{a} + \vec{b}, \vec{c}\right)$, we may use theorem 3.3 and 3.13 as well as definition 3.14 to obtain

$$\begin{aligned} \det\left(\vec{a} + \vec{b}, \vec{c}\right) &= \widehat{(\vec{a} + \vec{b})} \bullet \vec{c} \\ &= \left(\widehat{\vec{a}} + \widehat{\vec{b}}\right) \bullet \vec{c} = \widehat{\vec{a}} \bullet \vec{c} + \widehat{\vec{b}} \bullet \vec{c} \\ &= \det\left(\vec{a}, \vec{c}\right) + \det\left(\vec{b}, \vec{c}\right) \; . \end{aligned}$$

The last part of the proof is left as an exercise for the reader. ∎

If $\det\left(\vec{a}, \vec{b}\right) = 0$, then $\widehat{\vec{a}} \bullet \vec{b} = 0$. According to theorem 3.6, this means that $\widehat{\vec{a}}$ and $\vec{b}$ are orthogonal. If $\widehat{\vec{a}}$ and $\vec{b}$ are orthogonal, $\vec{a}$ and $\vec{b}$ must be parallel. This yields the following theorem:

Theorem 3.17

Given two vectors $\vec{a}$ and $\vec{b}$, we have

$$\det\left(\vec{a}, \vec{b}\right) = 0 \quad \Longleftrightarrow \quad \vec{a} \parallel \vec{b}\ .$$

So, the determinant allows us to determine whether two vectors are parallel.

It is also possible to interpret the determinant geometrically. The absolute value of the determinant turns out to be equal to the area of the parallelogram spanned by the two vectors. We have

Theorem 3.18

The area of the parallelogram spanned by the two vectors $\vec{a}$ and $\vec{b}$ is

$$P = \left|\det\left(\vec{a}, \vec{b}\right)\right|\ .$$

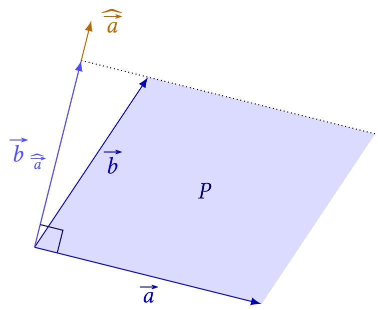


Figure 3.8: The parallelogram spanned by $\vec{a}$ and $\vec{b}$.

Proof
Figure 3.8 shows the parallelogram spanned by the two vectors $\vec{a}$ and $\vec{b}$. The base of the parallelogram is the vector $\vec{b}$, i.e. the length of the base is $|\vec{a}|$.

The figure also shows $\widehat{\vec{a}}$ and the projection $\vec{b}_{\widehat{\vec{a}}}$ of $\vec{b}$ onto this vector. The vector $\vec{b}_{\widehat{\vec{a}}}$ is perpendicular to $\vec{a}$, and its length corresponds to the distance between two parallel of the parallelogram. The area P must therefore be

$$P = \left|\vec{b}_{\widehat{\vec{a}}}\right| |\vec{a}|\ .$$

Using theorem 3.9, we rewrite this formula:

$$P = \frac{\left|\vec{b} \cdot \widehat{\vec{a}}\right|}{\left|\widehat{\vec{a}}\right|} |\vec{a}|\ .$$

But $\left|\widehat{\vec{a}}\right| = |\vec{a}|$, so this is also

$$P = \frac{\left|\vec{b} \cdot \widehat{\vec{a}}\right|}{|\vec{a}|} |\vec{a}| = \left|b \cdot \widehat{\vec{a}}\right| = \left|\det\left(\vec{a}, \vec{b}\right)\right|\ . \qquad \blacksquare$$

Example 3.19 Here, we calculate the area of the parallelogram spanned by the two vectors

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}\ .$$

According to theorem 3.18, the area is

$$\left|\det\left(\vec{a}, \vec{b}\right)\right| = \left\|\begin{matrix} 3 & 5 \\ 2 & 1 \end{matrix}\right\| = |3 \cdot 1 - 2 \cdot 5| = |-7| = 7\ .$$

So, the area of the parallelogram is 7.

Example 3.20 The two vectors

$$\vec{a} = \begin{pmatrix} 2 \\ t \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} ,$$

span a parallelogram with area 10. What is the value of the number t?

Theorem 3.18 allows us to write this expression for the area:

$$\left|\det\left(\vec{a}, \vec{b}\right)\right| = \left\|\begin{matrix} 2 & 1 \\ t & 4 \end{matrix}\right\| = |2 \cdot 4 - t \cdot 1| = |8 - t| \ .$$

But we know the area is 10, so we get the equation

$$|8 - t| = 10 \ .$$

This equation contains a numerical value, so we are actually dealing with two equations. If $8 - t$ equals 10, the equation is true; but this is also the case if $8 - t$ equals -10, i.e.

$$\begin{aligned} 8 - t = 10 \quad &\vee \quad 8 - t = -10 \qquad \Longleftrightarrow \\ t = -2 \quad &\vee \quad t = 18 \ . \end{aligned}$$

Thus, the area of the parallelogram is 10 if $t = -2$ or $t = 18$.

Until now, we have only interpreted geometrically the absolute value of the determinant. But the actual value of the determinant (with sign) is also linked to the geometry of the vectors.

Theorem 3.21

Let $\vec{a}$ and $\vec{b}$ be two vectors, and let θ be the angle from $\vec{a}$ to $\vec{b}$. Then

$$\det\left(\vec{a}, \vec{b}\right) = |\vec{a}|\left|\vec{b}\right|\sin(\theta) \ .$$

Proof

If θ is the angle from $\vec{a}$ to $\vec{b}$, the angle ϕ between $\widehat{\vec{a}}$ and $\vec{b}$ is

$$\phi = 90^\circ - \theta \ .$$

This relation between θ and ϕ may be found by analysing the positions of $\vec{a}$, $\widehat{\vec{a}}$, and $\vec{b}$, see figure 3.9.

Theorem 2.6 then implies

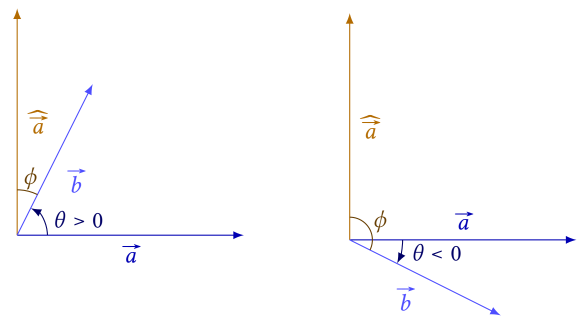
$$\cos(\phi) = \cos(90^\circ - \theta) = \sin(\theta) \ .$$

This means that[2]

$$\det(\vec{a}, \vec{b}) = \widehat{\vec{a}} \bullet \vec{b} = \left|\widehat{\vec{a}}\right|\left|\vec{b}\right|\cos(\phi) = |\vec{a}|\left|\vec{b}\right|\sin(\theta) \ .$$

This proves the theorem. ■

[2] In this calculation we use the fact that $\left|\widehat{\vec{a}}\right| = |\vec{a}|$, because a perpendicular vector has the same length as the original vector.



(a) $\phi = 90° - \theta$ (b) $\phi = 90° - \theta$

Figure 3.9: The relative position of the vectors $\vec{a}$ and $\vec{b}$ determines how we calculate ϕ, the angle between $\widehat{\vec{a}}$ and $\vec{b}$—but in every case we find $\phi = \theta - 90°$.

3.4 Exercises

Exercise 3.1
Calculate

a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ b) $\begin{pmatrix} -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ d) $\begin{pmatrix} 4 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Exercise 3.2
Given the three vectors

$$\vec{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} -3 \\ 4 \end{pmatrix},$$

calculate the angle between the vectors

a) $\vec{a}$ and $\vec{b}$ b) $\vec{a}$ and $\vec{c}$
c) $\vec{a} + \vec{b}$ and $\vec{c}$ d) $\vec{b} - \vec{c}$ and $\vec{a} + \vec{b}$
e) $\vec{a}$ and $\vec{b} - \vec{a}$ f) $\vec{c}$ and $\vec{a} + \vec{b} - \vec{c}$

Exercise 3.3
Determine whether the vectors are orthogonal:

a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Exercise 3.4
Given the three vectors

$$\vec{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} t \\ 2 \end{pmatrix},$$

determine the number t, such that

a) $\vec{a}$ and $\vec{c}$ are orthogonal
b) $\vec{b}$ and $\vec{c}$ are orthogonal
c) $\vec{a} + \vec{b}$ and $\vec{c}$ are orthogonal
d) $\vec{c}$ and $\vec{a} - \vec{c}$ are orthogonal

Exercise 3.5
Two vectors are given by

$$\vec{a} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Determine

a) $\vec{a}_{\vec{b}}$ b) $\vec{b}_{\vec{a}}$ c) $\vec{a}_{\vec{a}+\vec{b}}$

Exercise 3.6
Let three vectors be given:

$$\vec{a} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} t \\ 3 \end{pmatrix}.$$

a) Determine the length of the projection of $\vec{a}$ onto $\vec{b}$.

b) Determine t, such that $\vec{c}_{\vec{a}} = \vec{a}$.

c) Determine t, such that the length of the projection of $\vec{c}$ onto $\vec{b}$ is 2.

Exercise 3.7
Determine the perpendicular vector of $\vec{a} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, and draw $\vec{a}$ and $\widehat{\vec{a}}$ in a coordinate system.

Exercise 3.8
Calculate the followind determinants:

a) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$ b) $\begin{vmatrix} 5 & 1 \\ -4 & 7 \end{vmatrix}$

c) $\begin{vmatrix} 0 & 7 \\ 5 & -9 \end{vmatrix}$ d) $\begin{vmatrix} -4 & 3 \\ 7 & -2 \end{vmatrix}$

Exercise 3.9

The coordinates of the vectors $\vec{a}$ and $\vec{b}$ are

$$\vec{a} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} t \\ 2 \end{pmatrix} .$$

Determine the number t, such that

a) $\vec{a}$ and $\vec{b}$ are parallel

b) $\vec{a} + \vec{b}$ and $\widehat{\vec{b}}$ are parallel

Exercise 3.10

The coordinates of the three points P, Q and R are

$$P(3;2)\ , \quad Q(8,-1) \quad \text{and} \quad R(4;0)\ .$$

Determine the area of the parallelogram spanned by the vectors

a) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

b) $\overrightarrow{PR}$ and $\widehat{\overrightarrow{QR}}$

c) $2 \cdot \overrightarrow{PR}$ and $\widehat{\overrightarrow{RQ}}$

d) $\overrightarrow{PR} + \overrightarrow{QP}$ and $3 \cdot \overrightarrow{PQ}$

4 Triangles

A triangle may be described by three points, the *vertices* of the triangle. The triangle ABC is the triangle whose vertices are A, B and C. The side of the triangle from point A to point B is then called AB. The length of this side is denoted by $|AB|$.

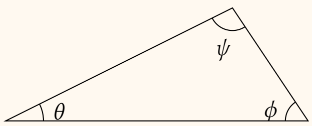
It is worth mentioning here that $|AB| = \left|\overrightarrow{AB}\right|$, i.e. the length of the side AB is the same as the length of the vector from A to B. Therefore, if we know the coordinates of the three points A, B, and C, we may calculate the lengths of the sides.

A triangle has three angles. The magnitude of these may also be found through vector calculations. First, however, we will look at some—hopefully well-know—properties of triangles.

Theorem 4.1

The sum of the angles of a triangle is 180°:

$$\theta + \phi + \psi = 180^\circ .$$



From each vertex of a triangle, we may draw the following lines:

A median which is a line from a vertex to the *midpoint* of the opposite side.

An angle bisector which is a line from a vertex to the opposite side, such that it *bisects (halves) the angle.*

An altitude from a vertex *perpendicular* to the opposite side. If the triangle is obtuse,[1] the altitude may fall outside the triangle.

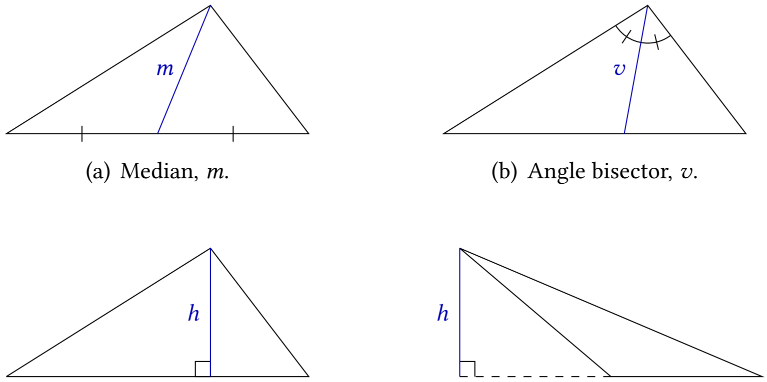
[1] A triangle is called *obtuse* if one of its angles is obtuse, i.e. greater than 90°.

These three lines are illustrated in figure 4.1.

An altitude is always perpendicular to the opposite side of the vertex. This opposite side is called the *base*,[2] and its length is used on the calculation of the area of the triangle.

[2] A triangle has three altitudes, which means there is more than one base. Which side we call the base depends on which vertex we are looking at.

We have the following well-known theorem connecting the area, altitude and base of a triangle:



(a) Median, m. (b) Angle bisector, v.

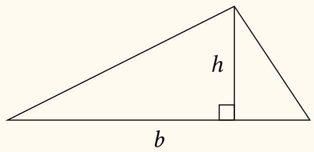
(c) Altitude, h. (d) Altitude outside the triangle.

Figure 4.1: Each vertex of a triangle has a corresponding median, angle bisector and altitude. Note that an altitude may fall outside the triangle.

Theorem 4.2

The area T of a triangle is half the altitude multiplied by the length of the corresponding base:

$$T = \frac{1}{2} \cdot h \cdot b \,.$$



4.1 Notation

When we talk about the sides and angles of a triangle, it is important to have an unambiguous notation, which clearly explains which quantity we are talking about.

We follow the convention illustrated in figure 4.2: The angle at the vertex A is called $\angle A$ etc., and the sides are denoted by the vertices they connect.

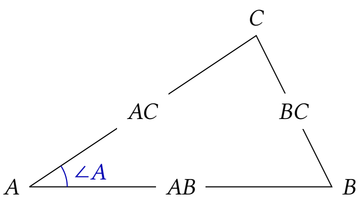


Figure 4.2: Sides and angles of $\triangle ABC$.

Sometimes, we also denote the lengths of the sides based on the *opposite* vertex. The length of the side opposite A is then denoted by a etc., such that

$$a = |BC| \;, \quad b = |AC| \;, \quad c = |AB| \;.$$

This is illustrated in figure 4.3.

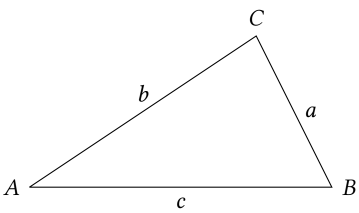


Figure 4.3: $\triangle ABC$ with side lengths denoted by a, b and c.

The lower-case notation for the lengths of the sides only applies when we look at exactly one triangle (i.e. a figure with exactly three vertices). If we look at more complicated figures with more than three vertices, several sides may be placed opposite a given vertex—the lower-case notation then becomes ambiguous. In this case, it is more sensible to write the lengths of the sides as $|AB|$, $|BC|$ etc.

If we look at figures with more than three points, sometimes situations arise where one letter is not enough to refer to an angle unambiguously, i.e. $\angle A$. In these cases, we denote an angle using 3 letters. E.g. $\angle CAD$ is the angle we get when we draw an angle from point C to point A to point D (see figure 4.4).

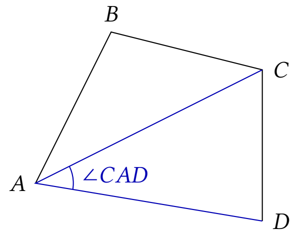


Figure 4.4: $\angle A$ may be 3 different angles in this figure; we therefore denote the marked angle by $\angle CAD$.

4.2 Similar Triangles

Two triangles with pairwise equal angles are called *similar* triangles, i.e.

Definition 4.3

If for two triangles ABC and $A'B'C'$ we have

$$\angle A' = \angle A, \quad \angle B' = \angle B \quad \text{og} \quad \angle C' = \angle C\,,$$

the two triangles are called *similar*.

If two triangles are similar, one of them is a larger or smaller copy of the other one. We have the following theorem:

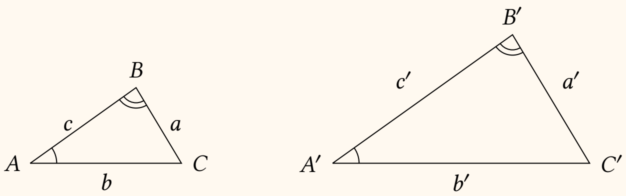
Theorem 4.4

If $\triangle ABC$ and $\triangle A'B'C'$ are similar, and

$$\angle A' = \angle A, \quad \angle B' = \angle B \quad \text{and} \quad \angle C' = \angle C\,,$$

the ratio between the lengths of the corresponding sides[3] are equal, i.e.

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}\,.$$

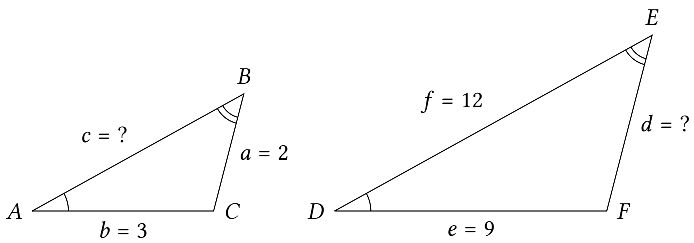


Example 4.5 If $\triangle ABC$ and $\triangle DEF$ are similar, and

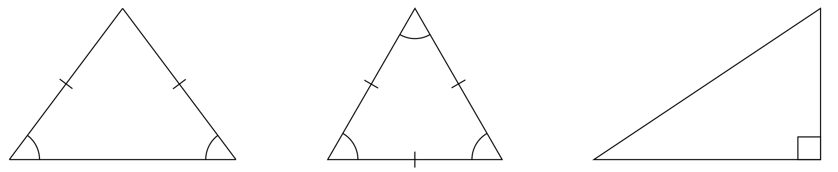
$$\angle A = \angle D, \quad \angle B = \angle E, \quad \text{and} \angle C = \angle F\,,$$

and we also know that $a = 2$, $b = 3$, $e = 9$ og $f = 12$, we may calculate the lengths of the remaining sides.

First, we draw a sketch (not necessarily to scale) to get an overview:



[3] The *corresponding* sides are the sides between equal angles.



(a) An isosceles triangle. (b) An equilateral triangle. (c) A right-angled triangle.

Figure 4.5: In an isosceles triangle, two sides are of equal length, in an equilateral triangle, all of the sides are equal, and in a right-angled triangle, one of the angles is a right angle.

Then we look at the ratios between the lengths of the corresponding sides. In this case,

$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c} \, .$$

Inserting the known values, we get

$$\frac{d}{2} = \frac{9}{3} = \frac{12}{c} \, .$$

I.e.

$$\frac{d}{2} = \frac{9}{3} \quad \Longleftrightarrow \quad d = \frac{9}{3} \cdot 2 = 6 \, ,$$

and

$$\frac{9}{3} = \frac{12}{c} \quad \Longleftrightarrow \quad \frac{c}{12} = \frac{3}{9} \quad \Longleftrightarrow \quad c = \frac{3}{9} \cdot 12 = 4 \, .$$

Now we know the length of every side of the two triangles.

4.3 Right-angled Triangles

Three special types of triangles are important. These are *isosceles* triangles, *equilateral* triangles, and *right-angled* triangles. These are illustrated in figure 4.5.

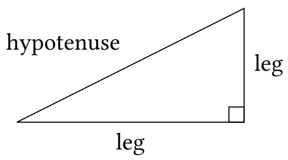


Figure 4.6: The sides of a right-angled triangle.

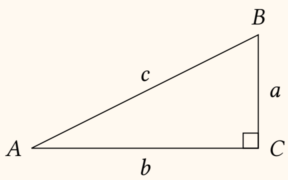
In a right-angled triangle, the sides also have special names. The side opposite the right angle is called the *hypotenuse*, while the two remaining sides are called *legs* (see 4.6).

The following theorem applies to right-angled triangles:

Theorem 4.6: Pythagorean theorem

In a right-angled triangle ABC, where $\angle C$ is the right angle,

$$a^2 + b^2 = c^2 \, .$$



If, instead of referring to a specific triangle ABC, we use the names of the sides of a right-angled triangle, the Pythagorean theorem may be written in the following way:

Theorem 4.7: Pythagorean theorem

In a right-angled triangle, the sum of the squares of the legs is equal to the square of the hypotenuse, i.e.

$$(\text{first leg})^2 + (\text{second leg})^2 = (\text{hypotenuse})^2 \ .$$

Example 4.8 If a triangle is right-angled, and we know that the length of one of the legs is 5, and the length of the hypotenuse is 13, we may calculate the length of the last leg using the Pythagorean theorem.

We call the length of the last leg x. We insert the known values into the Pythagorean theorem, and get the equation

$$5^2 + x^2 = 13^2 \ .$$

I.e.

$$x^2 = 13^2 - 5^2 = 169 - 25 = 144 \ .$$

Therefore, the length of the last leg is

$$x = \sqrt{144} = 12 \ .$$

If we draw an arbitrary right-angled triangle, the hypotenuse may be seen as a vector $\vec{c}$. Comparing this to the same right-angled triangles, where the side lengths are denoted by a, b, and c (see figure 4.7), allows us to derive these equations:

$$a = c \cdot \sin(\angle A) \quad \text{and} \quad b = c \cdot \cos(\angle A) \ .$$

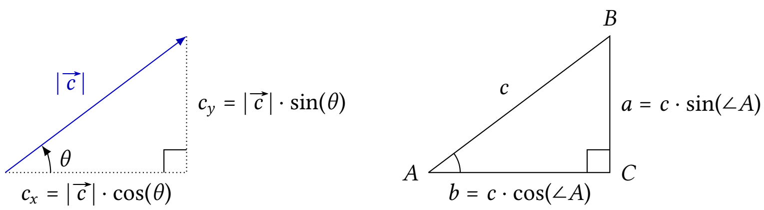
These may be rewritten, and we get

$$\sin(\angle A) = \frac{a}{c} \quad \text{and} \quad \cos(\angle A) = \frac{b}{c} \ .$$

From these formulas, we may derive a formula for $\tan(\angle A)$, since

$$\tan(\angle A) = \frac{\sin(\angle A)}{\cos(\angle A)} = \frac{a}{c} \Big/ \frac{b}{c} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} \ .$$

Summing up, we have the following theorem:



(a) Vector $\vec{c}$ as a hypotenuse.

(b) Side lengths a, b and c.

Figure 4.7: A vector may be drawn as the hypotenuse of a right-angled triangle. This allows us to derive a relation between the lengths of the sides and cos resp. sin of the directional angle of the vector.

Theorem 4.9

In a right-angled triangle ABC, where $\angle C$ is the right angle

$$\sin(\angle A) = \frac{a}{c}, \quad \cos(\angle A) = \frac{b}{c} \quad \text{and} \quad \tan(\angle A) = \frac{a}{b}.$$

Not every triangle is called ABC. Therefore, we sometimes write theorem 4.9 in the following way:

Theorem 4.10

In a right-angled triangle, where θ is one of the acute angles,

$$\sin(\theta) = \frac{\text{opposite leg}}{\text{hypotenuse}},$$
$$\cos(\theta) = \frac{\text{adjacent leg}}{\text{hypotenuse}},$$
$$\tan(\theta) = \frac{\text{opposite leg}}{\text{adjacent leg}}.$$

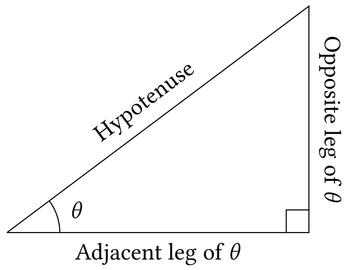
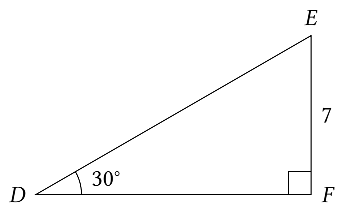


Figure 4.8: The names of the sides referring to the angle θ.

The labels "opposite" and "adjacent" refer to the leg's placement opposite or adjacent to the angle. See also figure 4.8.

Using the formulas in theorem 4.10, we may calculate the sides and angles in a right-angled triangle if we know at least one side and either another side or one of the acute angles.

Example 4.11 In a right-angled triangle DEF, $\angle D = 30°$ and $|EF| = 7$. A sketch of the triangle looks like this:



The side EF is the opposite leg of $\angle D$, and DE is the hypotenuse, so according to theorem 4.10

$$\sin(30°) = \frac{7}{|DE|}.$$

We solve this equation and get

$$|DE| = \frac{7}{\sin(30°)} = 14.$$

We may now determine the last side using the Pythagorean theorem, and the last angle from the sum of the angles of the triangles.

Example 4.12 If we want to determine the length of the leg DF in the triangle from example 4.11, we may do so using the tangent.

DF is the adjacent leg of $\angle D$, and EF is the opposite leg, so, theorem 4.10 yields

$$\tan(30^\circ) = \frac{7}{|DF|} \,.$$

Solving this equation, we get

$$|DF| = \frac{7}{\tan(30^\circ)} = 12.1 \,.$$

4.4 The Area of a Triangle

In this section and the next, we use vector geometry to derive some theorems about triangles. First, we examine how the determinant of two vectors may be used to determine the area of a triangle.

If the vertices of a triangle are located at the points A, B, and C, the area of the triangle is half the area of the parallelogram spanned by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore, theorem 3.18 leads to the following theorem about the areas of triangles:

Theorem 4.13

The area of the triangle with vertices at the points A, B, and C is

$$T = \tfrac{1}{2}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right| \,.$$

Example 4.14 The points $A(-1,3)$, $B(0,5)$ og $C(7,2)$ are the vertices of a triangle. The area of this triangle may be determined using theorem 4.13.

First, we find the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We get

$$\overrightarrow{AB} = \begin{pmatrix} 0-(-1) \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\overrightarrow{AC} = \begin{pmatrix} 7-(-1) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix} \,.$$

We may now calculate the area of the triangle ABC:

$$T = \tfrac{1}{2}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right| = \tfrac{1}{2}\left\|\begin{matrix} 1 & 8 \\ 2 & -1 \end{matrix}\right\| = \tfrac{1}{2}\left|1\cdot(-1) - 2\cdot 8\right| = \tfrac{1}{2}\left|-17\right| = \tfrac{17}{2} \,.$$

So, the area of the triangle ABC is $\frac{17}{2}$.

Theorem 4.13 combined with theorem 3.21 shows that the area of a triangle ABC is

$$T = \tfrac{1}{2}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right| = \tfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\left|\sin(\theta)\right| \,, \tag{4.1}$$

where θ is the angle from $\overrightarrow{AB}$ to $\overrightarrow{AC}$. If the lengths of the sides of the triangle are denoted by a, b, and c, (4.1) may be rewritten to provide the following theorem:

Theorem 4.15

The area of triangle ABC is

$$T = \tfrac{1}{2} b\, c \sin(\angle A)\ .$$

The formula in theorem 4.15 calculates the area of a triangle from two sides and the angle between them. Therefore, this formula may be written in three different ways, depending on which angle we use as a starting point:

$$\begin{aligned} T &= \tfrac{1}{2} b\, c \sin(\angle A) \\ T &= \tfrac{1}{2} a\, c \sin(\angle B) \\ T &= \tfrac{1}{2} a\, b \sin(\angle C) \end{aligned}$$

4.5 The Law of Sines

From theorem 4.15 we may derive the following:

Theorem 4.16: The law of sines

In a given triangle ABC

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}\ ,$$

and

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}\ .$$

Proof

As mentioned above, the formula in theorem 4.15 may be written as three formulas for the area of the same triangle. Therefore, we must have

$$\tfrac{1}{2} \cdot b \cdot c \cdot \sin(\angle A) = \tfrac{1}{2} \cdot a \cdot c \cdot \sin(\angle B) = \tfrac{1}{2} \cdot a \cdot b \cdot \sin(\angle C)\ .$$

In this double equation, we divide by $\frac{1}{2} \cdot a \cdot b \cdot c$ on all "sides", and get

$$\frac{\frac{1}{2} \cdot b \cdot c \cdot \sin(\angle A)}{\frac{1}{2} \cdot a \cdot b \cdot c} = \frac{\frac{1}{2} \cdot a \cdot c \cdot \sin(\angle B)}{\frac{1}{2} \cdot a \cdot b \cdot c} = \frac{\frac{1}{2} \cdot a \cdot b \cdot \sin(\angle C)}{\frac{1}{2} \cdot a \cdot b \cdot c}\ .$$

We now reduce this as much as possible, and we are left with

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}\ ,$$

which proves the theorem. ■

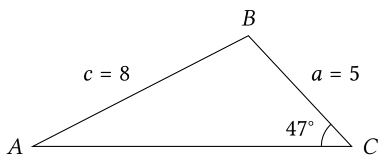


Figure 4.9: A triangle with two known sides, and one known angle.

The law of sines states that the ratio between the sine of an angle and the length of the opposite side is constant in any given triangle. If we know an angle and the length of the opposite side, and we know either another angle or another length, we may use this to calculate the unknown side lengths and angles of the triangle.

Example 4.17 In triangle ABC, $\angle C = 47°$, $a = 5$ and $c = 8$. Figure 4.9 shows a sketch of this triangle

According to the law of sines (theorem 4.16), we have

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle C)}{c} ,$$

i.e.

$$\frac{\sin(\angle A)}{5} = \frac{\sin(47°)}{8} \qquad \Longleftrightarrow \qquad \sin(\angle A) = \frac{\sin(47°)}{8} \cdot 5 = 0.4571 .$$

Therefore,

$$\angle A = \sin^{-1}(0.4571) = 27.2° .$$

We may now determine $\angle B$ from the sum of the angles of the triangle, and the lengh of the last side may also be found using the law of sines (see next example).

Example 4.18 In $\triangle ABC$, $\angle A = 62°$, $\angle B = 34°$, and $b = 7$. Figure 4.10 shows a sketch of the triangle.

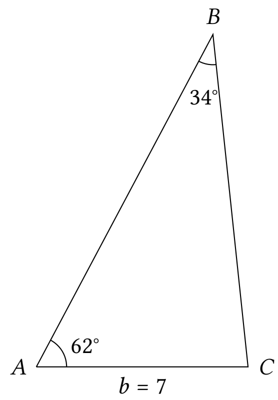


Figure 4.10: A triangle with two known angles, and one known side.

According to the law of sines

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} ,$$

i.e.

$$\frac{a}{\sin(62°)} = \frac{7}{\sin(34°)} \qquad \Longleftrightarrow \qquad a = \frac{7}{\sin(34°)} \cdot \sin(62°) = 11.1 .$$

We may use the angle sum to calculate the last angle, and the length of the last side may be determined in the same way as the side a.

The ambiguous case

It turns out that we need to be extra careful whenever we use the law of sines to calculate angles. The function $\sin^{-1}$ which we use to calculate angles, always yields a result between $-90°$ and $90°$; but in a triangle, an angle may be up to 180°—and for any given value of the sine, two angles are possible.

Figure 4.11 shows the unit vector $\vec{e}_1$ with directional angle θ, and the unit vector $\vec{e}_2$ with directional angle $180° - \theta$. The figure shows that these two unit vectors have the same y-coordinate, i.e.

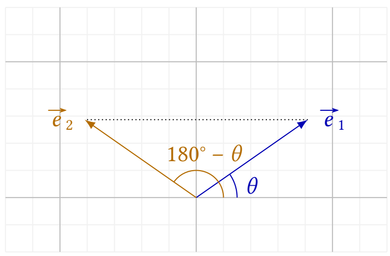


Figure 4.11: Two angles with the same sine.

$$\sin(180° - \theta) = \sin(\theta) .$$

This shows that the equation $\sin(\theta) = y$ may have two solutions. The figure also shows that the solutions are

$$\theta = \sin^{-1}(y) \qquad \text{and} \qquad \theta = 180° - \sin^{-1}(y) .$$

Example 4.19 In triangle ABC, $\angle A = 56°$, $a = 7$, and $b = 8$. Using the law of sines we calculate $\angle B$. We get

$$\frac{\sin(\angle B)}{b} = \frac{\sin(\angle A)}{a} .$$

This gives us the equation

$$\frac{\sin(\angle B)}{8} = \frac{\sin(56°)}{7} \qquad \Longleftrightarrow \qquad \sin(\angle B) = \frac{\sin(56°)}{7} \cdot 8 = 0.9475 .$$

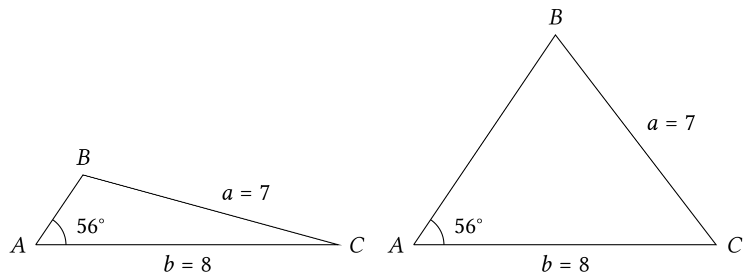
This equation has two solutions. One is[4]

$$\angle B = \sin^{-1}(0.9475) = 71.3° ,$$

and the other is

$$\angle B = 180° - \sin^{-1}(0.9475) = 108.7° .$$

Therefore, triangle ABC may look like either of these two triangles:



If we want to determine the remaining sides and angles of $\triangle ABC$, we need to do the calculations for two different triangles. Therefore, there is not one, but two solutions—and neither of the is more correct than the other.

[4]An equation such as $\sin(\angle B) = 0.9475$ has two solutions exactly because $\sin(71.3°) = \sin(108.7°)$, so, both angles solve the equation.

Even though the equation $\sin(\theta)$ always has two solutions, both solutions need not make sense. This is demonstrated in the next example.

Example 4.20 Here, we look at the triangle ABC where $\angle A = 45°$, $a = 15$, and $b = 12$. The law of sines states that

$$\frac{\sin(\angle B)}{b} = \frac{\sin(\angle A)}{a} ,$$

which yields the equation

$$\frac{\sin(\angle B)}{12} = \frac{\sin(45°)}{15} \qquad \Longleftrightarrow \qquad \sin(\angle B) = \frac{\sin(45°)}{15} \cdot 12 = 0.5657 .$$

This equation has two solutions,

$$\angle B = \sin^{-1}(0.5657) = 34.4°$$

and

$$\angle B = 180° - \sin^{-1}(0.5657) = 145.6° .$$

The last solution *is* a solution to the equation $\sin(\angle B) = 0.5657$, but it is a meaningless solution in the given context. The sum of the angles of a triangle is 180°, and we already have an angle of 45°. Then we cannot also have an angle of 145.6°, wherefore we discard this solution.

So, there is only one possible angle which fits, and $\angle B = 34.4°$.

4.6 The Law of Cosines

We may use the law of sines in the cases where we know an angle and a side opposite each other. In cases where we do not know such a pair, we may only calculate remaining sides and angles by using the *law of cosines*.

Theorem 4.21: The law of cosines

In any given triangle ABC, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\angle A) & \cos(\angle A) &= \frac{b^2 + c^2 - a^2}{2 \cdot b \cdot c} \\ b^2 &= a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos(\angle B) & \cos(\angle B) &= \frac{a^2 + c^2 - b^2}{2 \cdot a \cdot c} \\ c^2 &= a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\angle C) & \cos(\angle C) &= \frac{a^2 + b^2 - c^2}{2 \cdot a \cdot b} \,. \end{aligned}$$

The formulas on the right hand side of theorem 4.21 are merely a rewritten version of the formulas on the left.

If we take a closer look at the formulas on the left hand side, we notice that every formula enables us to calculate the length of a side when we know the opposite angle and the other two sides. The content of all three formulas is the same, so, we only need to prove one of them.

Proof
The triangle ABC is spanned by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. According to theorem 1.13, we also have

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AB} \,.$$

Since $a^2 = \left|\overrightarrow{BC}\right|^2$, we may find an expression for a^2 by examining the vector $\overrightarrow{BC}$. Using the rules from theorem 3.3 og sætning 3.4, we get

$$\begin{aligned} \left|\overrightarrow{BC}\right|^2 &= \left|\overrightarrow{AC} - \overrightarrow{AB}\right|^2 = \left(\overrightarrow{AC} - \overrightarrow{AB}\right) \bullet \left(\overrightarrow{AC} - \overrightarrow{AB}\right) \\ &= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AB}\right|^2 - 2 \cdot \overrightarrow{AC} \bullet \overrightarrow{AB} \\ &= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AB}\right|^2 - 2 \cdot \left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \cos(\angle A) \\ &= b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\angle A) \,, \end{aligned}$$

which proves the theorem. ■

The law of cosines may be used to calculate the length of a side when we know the two other sides of the triangle, and the opposite angle. This corresponds to the formulas on the left hand side of theorem 4.21.

Alternatively, we may use the law of cosines to calculate an angle when all three sides of the triangle are known—this corresponds to the formulas on the right hand side of the theorem.

Example 4.22 In triangle ABC, $\angle C = 39°$, $a = 7$, and $b = 10$. Figure 4.12 shows a sketch of the triangle.

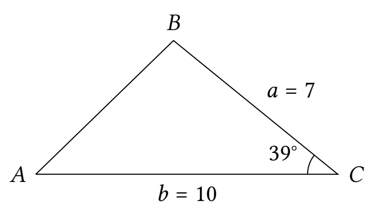


Figure 4.12: Triangle where two sides and the angle in between is know.

The length of the side c may now be found using the law of cosines. According to theorem 4.21

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\angle C) = 7^2 + 10^2 - 2 \cdot 7 \cdot 10 \cdot \cos(39°) = 40.20 \ ,$$

so,

$$c = \sqrt{40.20} = 6.3 \ .$$

Now that we know the lengths of all the sides, one of the remaining angles may also be found using the law of cosines (see next example). Alternatively, we could determine one of the angles using the law of sines.

Example 4.23 In this example, we look at a triangle ABC where $a = 3$, $b = 6$, and $c = 4$ (see figure 4.13). According to the law of cosines, we may determine $\angle A$ by using the formula

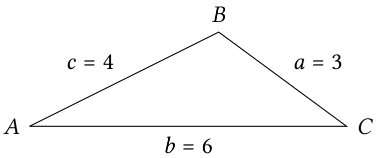


Figure 4.13: Triangle where all the sides are known.

$$\cos(\angle A) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{6^2 + 4^2 - 3^2}{2 \cdot 6 \cdot 4} = 0.8958 \ .$$

This gives us

$$\angle A = \cos^{-1}(0.8958) = 26.4° \ .$$

The remaining angles may be found in a similar way.

4.7 Exercises

Exercise 4.1
Determine the area of triangle PQR when $|PQ| = 6$ and $h_R = 3$.

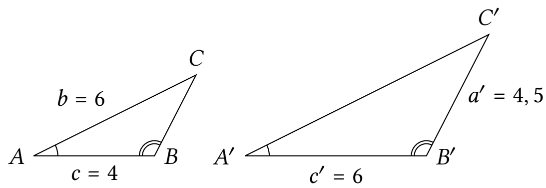
Exercise 4.2
The area of triangle ABC is 28, and $h_C = 7$.

a) Calculate $|AB|$.

Exercise 4.3
The area of triangle ABC is 12, and $|BC| = 6$.

a) Determine the altitude h_A.

Exercise 4.4
The two triangles in the figure are similar.

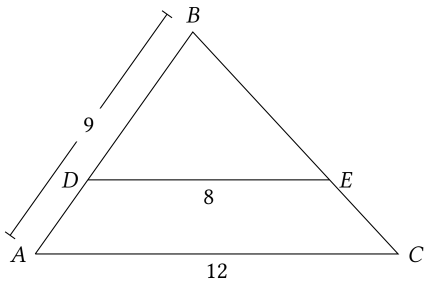


a) Determine the side lengths a and b'.

Exercise 4.5
The two triangles ABC and DEF are similar, such that $\angle A = \angle D$ and $\angle B = \angle E$. We also have $a = 5$, $b = 7$, $e = 15$, and $f = 18$.

a) Determine the lengths of the remaining sides.

Exercise 4.6
In the figure below, AC and DE are parallel. Some measurements are shown in the figure.



a) Determine $|AD|$.

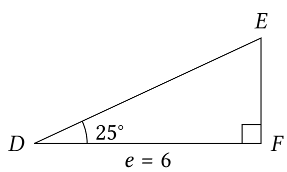
Exercise 4.7
Triangle ABC right-angled, and $\angle C$ is the right angle.

a) Determine c when $a = 3$ and $b = 4$.
b) Determine a when $b = 5$ and $c = 13$.

Exercise 4.8
Triangle BLT is a right-angled triangle where $\angle L$ is the right angle.

a) Write the Pythagorean theorem for this triangle.
b) Determine b when $l = 4.3$ and $t = 2.9$.

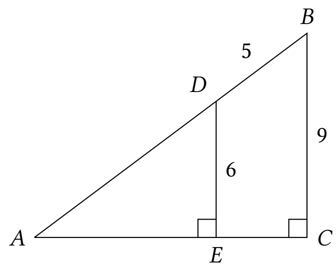
Exercise 4.9
The right-angled triangle DEF is depicted below. Some of the triangle's measurements are shown in the figure.



a) Determine the remaining angle and the remaining sides of triangle DEF.
b) Determine the area of the triangle.

Exercise 4.10
Determine the remaining angles and the length of the last leg in a right-angled triangle where the length of one leg is 3, and the length of the hypotenuse is 7.

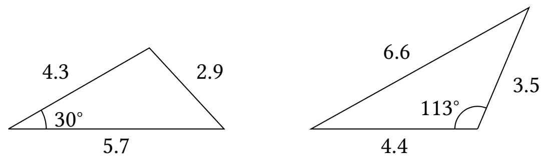
Exercise 4.11
In the figure below, $|DE| = 6$, $|BC| = 9$, and $|BD| = 5$.



a) Determine $|AD|$ and $|AE|$.
b) Determine $\angle A$ and $\angle B$.

Exercise 4.12
Determine the area of triangle ABC when $a = 3$, $b = 5$, and $\angle C = 72°$.

Exercise 4.13
Determine the areas of the triangles below.



Exercise 4.14
In triangle ABC, $\angle A = 26°$, $\angle B = 95°$, and $|BC| = 4.0$.

a) Determine the remaining sides and the remaining angle of the triangle.
b) Determine the area of the triangle.

Exercise 4.15
Determine the remaining quantities of triangle ABC in the four cases below. Remember in each case to examine whether more than one solution exists.

a) $\angle A = 34°$, $a = 10$, $b = 5$.
b) $\angle B = 69°$, $\angle C = 71°$, $a = 4.7$.
c) $\angle B = 106°$, $b = 5$, $c = 6.2$.
d) $\angle A = 62°$, $\angle B = 79°$, $c = 8.3$.

Exercise 4.16
In triangle ABC, $\angle A = 32°$, $c = 4.7$, and the length of the median from B is $m_B = 6.1$.

a) Determine b.
b) Determine the area of the triangle.

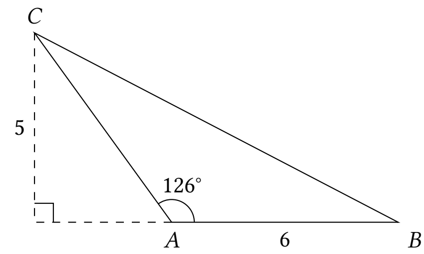
Exercise 4.17
Determine the remaining sides and angles of triangle ABC in the four cases below.

a) $a = 5$, $b = 7$, $\angle C = 49°$.
b) $a = 3$, $b = 2$, $c = 4$.
c) $b = 4$, $c = 7$, $\angle A = 112°$.
d) $a = 9$, $b = 5$, $c = 7$.

Exercise 4.18
The vertices of the triangle ABC are at the points $A(-2, 3)$, $B(0, 7)$, and $C(4, 1)$.

a) Determine the lengths of the sides and the angles of the triangle.
b) Determine the area of the triangle.
c) Determine the altitude h_C from C onto the side AB.

Exercise 4.19
In triangle ABC, $\angle A = 126°$, $|AB| = 6$, and the length of the altitude from C is $h_C = 5$.



a) Determine the area of the triangle.
b) Determine the length of AC.
c) Determine $|BC|$ as well as $\angle B$ and $\angle C$.

Lines 5

In this chapter, we look at lines in the plane. Using points and vectors, we will derive equations for the points on a given line. In this context, a formula for the *distance* between to points will be useful. We have the following theorem:

Theorem 5.1

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Proof
The distance from A to B equals the length of the vector $\overrightarrow{AB}$, i.e.

$$|AB| = \left|\overrightarrow{AB}\right| = \left|\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ . \qquad \blacksquare$$

We may also prove the theorem by viewing the line AB as a hypotenuse in a right-angled triangle and applying the Pythagorean theorem. See figure 5.1.

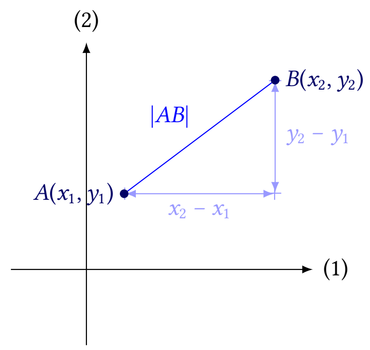


Figure 5.1: We may find the distance between two points by using the Pythagorean theorem.

5.1 Parametrisation of a Line

A line in a coordinate system may be described by a point on the line and the direction of the line in the coordinate system. When we use the direction to describe the line, we use a vector to describe the direction.

Definition 5.2

A vector $\vec{r}$ parallel to a line l is called a *direction vector* of l.

A vector $\vec{n}$ perpendicular to a line l is called a *normal vector* of l.

Here, tt is important to note that a line does not have one direction vector but infinitely many. If $\vec{r}$ is a direction vector of l, any vector parallel to $\vec{r}$ is also a direction vector. The same reasoning applies to normal vectors.

When we desribe the line it is also unimportant at which point we begin. A line contains an infinity of points, and any one of these combined with a direction vector describes the line.

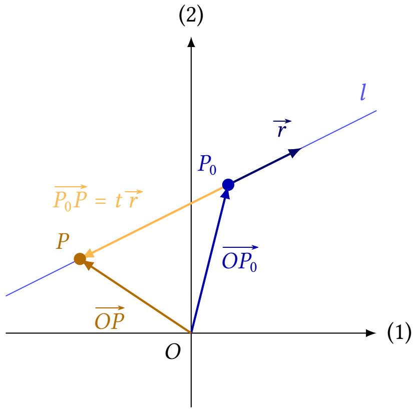


Figure 5.2: From this figure it is possible to derive a relation between a random point P on the line, the known point P_0, and the direction vector $\vec{r}$.

If we have a point and a direction vector of a line, the line may be described by a so-called *parametrisation* or *parametric representation*, which is a set of equations describing how to calculate the coordinates of each point on the line via a starting point and a direction vector.

Figure 5.2 shows a line l, a point $P(x_0, y_0)$ on the line, and a direction vector $\vec{r}$ of the line. The figure also shows a random point $P(x, y)$ on the line. According to theorem 1.13, we have

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}\ .$$

This is also illustrated in the figure.

But the vector $\overrightarrow{P_0P}$ is parallel to $\vec{r}$, i.e. a number t exists, so that $\overrightarrow{P_0P} = t\vec{r}$. Therefore we arrive at the equation

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{r}\ .$$

$\overrightarrow{OP}$ and $\overrightarrow{OP_0}$ are position vectors of the two points $P(x, y)$ og $P_0(x_0, y_0)$, so, these vectors have the same coordinates as the two points. We may insert these into the equation and get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\vec{r}\ . \tag{5.1}$$

Any point $P(x, y)$ in the line may be found through this equation for some value of t. If we let t run through the set of real numbers, we get every point on the line. An equation of the type (5.1) where $t \in \mathbb{R}$ is called a *parametrisation* of the line l. The number t, which runs through the real numbers, is called the *parameter*. Thus, we have

Theorem 5.3

A *parametrisation* of a line in the plane is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} r_x \\ r_y \end{pmatrix}\ , \qquad t \in \mathbb{R}\ ,$$

where (x_0, y_0) is a point on the line, and $\vec{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix}$ is a direction vector of the line.

Example 5.4 If a line l passes through the point $(3, 2)$ and has direction vector $\vec{r} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, then a parametrisation of the line is

$$l:\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 5 \end{pmatrix}\ , \qquad t \in \mathbb{R}\ .$$

Example 5.5 Here, we determine a parametrisation of the line m which passes through the two points $A(3, 4)$ and $B(-2, 6)$.

To write a parametrisation, we need a direction vector. Both A and B are on the line, so, $\overrightarrow{AB}$ is a direction vector.

$$\overrightarrow{AB} = \begin{pmatrix} -2-3 \\ 6-4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}\ .$$

We also need a point on the line. Here, the obvious choice is either A or B. If we use the point A, we get the parametrisation

$$m: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

The points A and B, the line m, and the direction vector $\overrightarrow{AB}$ are illustrated in figure 5.3.

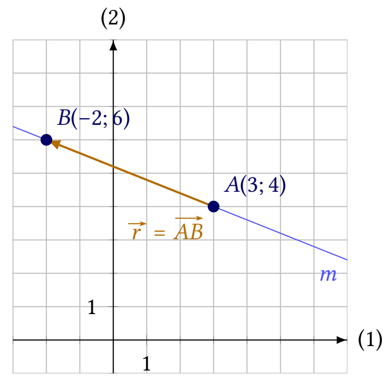


Figure 5.3: The points A and B, and the line m.

Since m passes through an infinity of points and has infinitely many direction vectors (every vector parallel to $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$), e.g.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} -10 \\ 4 \end{pmatrix}$$

are also parametrisations of m.

A parametrisation is actually a vector equation, where the coordinates of the direction vector $\begin{pmatrix} x \\ y \end{pmatrix}$ are functions of the parameter t. This means that given the parametrisation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} r_x \\ r_y \end{pmatrix}, \quad t \in \mathbb{R},$$

we can write the two *coordinate functions* $x(t)$ and $y(t)$, which are

$$x(t) = x_0 + r_x t \qquad \text{and} \qquad y(t) = y_0 + r_y t.$$

These two functions shows how the x- and y-coordinates of the point change as functions of the parameter t.

Therefore, the parameter t in the parametrisation is often interpreted as time. I.e. as the time t passes, the point $(x(t), y(t))$ moves along the curve given by the parametrisation.[1]

[1] Parametrisations exist for infinitely many different types of curves, e.g. circles or parabolas.

If two lines are not parallel, they intersect. We may use the coordinate functions to determine the intersection point between the two lines.

Example 5.6 Two lines are given by the parametrisations[2]

[2] When we write more than one parametrisation, it is important to denote the parameters by different letter, because the parameters represent different numbers.

$$l: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 6 \end{pmatrix}, \quad t \in \mathbb{R}$$
$$m: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \quad s \in \mathbb{R}$$

For the line l, the coordinate functions

$$x_l = 3 + t \cdot (-1) \qquad \text{and} \qquad y_l = -3 + t \cdot 6,$$

and for m, we get

$$x_m = -4 + s \cdot (5) \qquad \text{and} \qquad y_m = 1 + s \cdot 8.$$

To determine the intersection point, we need to find the value of t and the value of s, which yield the same point when we insert them into the equations. I.e. values of t and s where $x_l = x_m$ and $y_l = y_m$. Thus, we have two equations in two unknowns:

$$\begin{aligned} 3 - t &= -4 + 5s \\ -3 + 6t &= 1 + 8s \ . \end{aligned} \tag{5.2}$$

We solve these two equations by the method of elimination. Multiplying the top equation by 6 yields

$$\begin{aligned} 18 - 6t &= -24 + 30s \\ -3 + 6t &= 1 + 8s \ . \end{aligned}$$

Adding these two equations eliminates the parameter t; we get the new equation

$$15 = -23 + 38s \qquad \Longleftrightarrow \qquad s = 1 \ .$$

Now that we know the parameter s, we may determine the intersection point from the parametrisation of m. Inserting $s = 1$ into the parametrisation, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} \ .$$

So, the two lines intersect at $(1, 9)$.

If we want to confirm this calculation, we may calculate the value of the parameter t, e.g. via the equation (5.2). Inserting $s = 1$ into this equation, we get

$$3 - t = -4 + 5 \cdot 1 \qquad \Longleftrightarrow \qquad t = 2 \ .$$

When we let $t = 2$ in the parametrisation of l, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} \ ,$$

which merely confirms that the intersection point of the two lines is $(1, 9)$.

5.2 Equation of the Line

Lines may also be described by equations. Therefore, it is always possible to derive an equation of a line from a parametrisation. E.g., assume that a given line l has the parametrisation

$$l : \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\vec{r} \ , \quad t \in \mathbb{R} \ .$$

Then the parametrisation may be rewritten as

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = t\vec{r} \ . \tag{5.3}$$

The perpendicular vector $\widehat{\vec{r}}$ of the direction vector $\vec{r}$ is perpendicular to the line. Therefore, this is a normal vector of the line. The coordinates of this normal vector may be calculated from the coordinates of $\vec{r}$. Below, these coordinates are denoted by a and b, i.e. $\widehat{\vec{r}} = \begin{pmatrix} a \\ b \end{pmatrix}$.

We calculate the scalar product by this vector on both sides of equation (5.3) and get

$$\widehat{\vec{r}} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \widehat{\vec{r}} \bullet t\vec{r} ,$$

but because $\widehat{\vec{r}} \perp \vec{r}$, the right hand side is 0, i.e. we get

$$\begin{aligned} \widehat{\vec{r}} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} &= 0 \quad \iff \\ \begin{pmatrix} a \\ b \end{pmatrix} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} &= 0 \quad \iff \\ a(x - x_0) + b(y - y_0) &= 0 . \end{aligned}$$

Thus, we have the following theorem:

Theorem 5.7

A line passing through the point (x_0, y_0) with normal vector $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ may be described by the equation

$$a(x - x_0) + b(y - y_0) = 0 ,$$

which we may also write as

$$ax + by + c = 0 ,$$

where $c = -ax_0 - by_0$.

Example 5.8 The equation of the line through $(3, 1)$ with normal vector $\vec{n} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ is

$$\begin{aligned} 5(x - 3) + (-2)(y - 1) &= 0 \quad \iff \\ 5x - 15 - 2y + 2 &= 0 \quad \iff \\ 5x - 2y - 13 &= 0 . \end{aligned}$$

Example 5.9 We may rewrite the following parametrisation

$$l : \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \end{pmatrix} , \quad t \in \mathbb{R}$$

as an equation of a line. First, we find a normal vector. We use the perpendicular vector of the direction vector of the line, i.e.

$$\vec{n} = \widehat{\begin{pmatrix} 4 \\ -3 \end{pmatrix}} = \begin{pmatrix} -(-3) \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} .$$

The line passes through the point $(-1, 5)$. So, the equation of the line is

$$l : \quad 3(x - (-1)) + 4(y - 5) = 0 ,$$

which we reduce to

$$l: \quad 3x + 4y - 17 = 0 \,.$$

It is important to note that the number a in the equation $ax + by + c = 0$ is *not a slope*. However, if $b \neq 0$, we may rewrite the equation:

$$ax + by + c = 0 \quad \Longleftrightarrow \quad y = -\frac{a}{b}x - \frac{c}{b} \,.$$

i.e. the slope of the line is $-\frac{a}{b}$. If $b = 0$ this, it is impossible to rewrite the equation this way, because we cannot divide by 0. If $b = 0$, the equation of the line has the form

$$ax + c = 0 \quad \Longleftrightarrow \quad x = -\frac{c}{a} \,,$$

i.e. the line is parallel to the y-axis (a "vertical" line).

Therefore, the form $ax + by + c = 0$ describes *any* line in a coordinate system, even the vertical lines—which are impossible to describe by a line of the form $y = \alpha x + \beta$,[3]

[3] The slope and y-axis intercept of the line are here called α and β, so as not to confuse them with the contants a and b in the equation $ax + by + c = 0$.

From the argument presented above, we derive the theorem

Theorem 5.10

If a line is given by the equation

$$ax + by + c = 0 \,,$$

where $b \neq 0$, we may write the equation as

$$y = \alpha x + \beta \,,$$

where $\alpha = -\frac{a}{b}$ is the slope, and $\beta = -\frac{c}{b}$ is the y-axis intercept.

If $b = 0$, the line is parallel to the y-axis, and the equation cannot be rewritten this way.

Using both of these forms, we arrive at the following theorem about orthogonal lines:

Theorem 5.11

The two lines l and m with equations

$$\begin{aligned} l: &\quad y = \alpha_1 x + \beta_1 \\ m: &\quad y = \alpha_2 x + \beta_2 \end{aligned}$$

are orthogonal, exactly when $\alpha_1\alpha_2 = -1$.

Proof

We write the two equations in the form $ax + by + c = 0$:

$$\begin{aligned} l: &\quad \alpha_1 x - y + \beta_1 = 0 \\ m: &\quad \alpha_2 x - y + \beta_2 = 0 \,. \end{aligned}$$

We now see that the normal vectors of the two lines are

$$\vec{n}_l = \begin{pmatrix} \alpha_1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{n}_m = \begin{pmatrix} \alpha_2 \\ -1 \end{pmatrix} .$$

If the two lines are orthogonal, their normal vectors are also orthogonal (see figure 5.4), i.e.

$$\begin{aligned} \vec{n}_l \bullet \vec{n}_m &= 0 &&\iff \\ \begin{pmatrix} \alpha_1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ -1 \end{pmatrix} &= 0 &&\iff \\ \alpha_1 \alpha_2 + (-1) \cdot (-1) &= 0 &&\iff \\ \alpha_1 \alpha_2 + 1 &= 0 &&\iff \\ \alpha_1 \alpha_2 &= -1 . \end{aligned}$$

On the other hand, if we know that $\alpha_1 \alpha_2 = -1$, we may follow the derivation above backwards and arrive at the conclusion that if the normal vectors are orthogonal, then so are the lines. ■

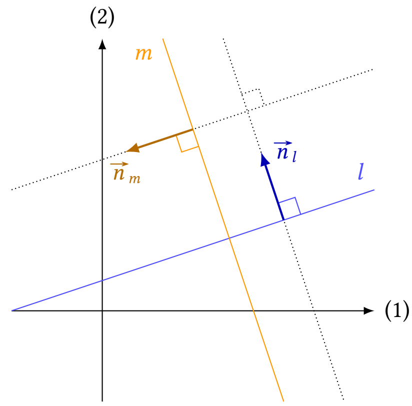


Figure 5.4: If two lines are orthogonal, then so are their normal vectors.

Example 5.12 Here, we determine the line m through $(8, 1)$ which is perpendicular to the line

$$l : \quad y = -4x + 3 .$$

If the two lines are orthogonal, the product of their slopes is -1. If we denote the slope of m by α ,we have

$$\alpha \cdot (-4) = -1 \quad \iff \quad \alpha = \frac{1}{4} .$$

The line m passes through the point $(8, 1)$, so, the equation of the line is

$$y = \tfrac{1}{4}(x - 8) + 1 ,$$

which we reduce to

$$m : \quad y = \tfrac{1}{4}x - 1 .$$

The two lines and the point are illustrated in figure 5.5.

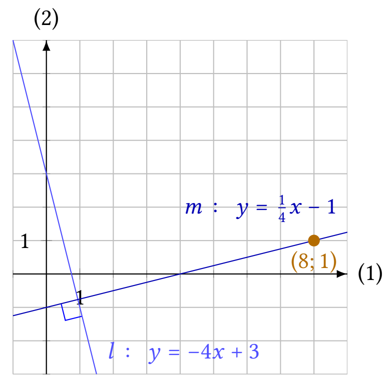


Figure 5.5: The two lines l and m are orthogonal.

5.3 Distance from a Point to a Line

If we have a line in the plane, and a point, which is not on the line, it would be useful to be able to calculate the distance between the point and the line. If by distance, we mean the *shortest* distance between the point and the line, we have the following theorem:

Theorem 5.13

The distance form the point $P(x_0, y_0)$ to the line $l : \; ax + by + c = 0$ is

$$\operatorname{dist}(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} .$$

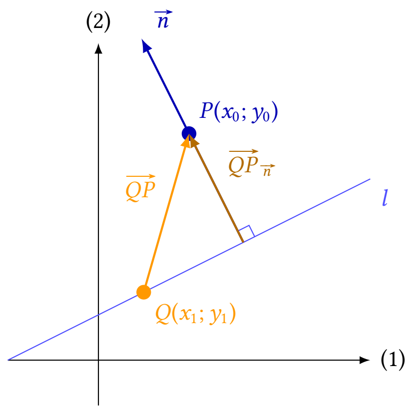


Figure 5.6: The distance from the point to the line equals the length of the vector $\overrightarrow{QP}_{\vec{n}}$.

Proof

Figure 5.6 shows the line l and the point P. The figure also shows the normal vector $\vec{n}$ of the line, and a random point $Q(x_1, y_1)$ on the line.

If we project the vector $\overrightarrow{QP}$ onto the normal vector $\vec{n}$, we get a vector $\overrightarrow{QP}_{\vec{n}}$, the length of which is the distance we are looking for. Using theorem 3.9, we now calculate the length of this vector:

$$\operatorname{dist}(P, l) = \left|\overrightarrow{QP}_{\vec{n}}\right| = \frac{\left|\overrightarrow{QP} \bullet \vec{n}\right|}{|\vec{n}|} . \tag{5.4}$$

If the line has equation $ax + by + c = 0$, and the points have the coordinates $P(x_0, y_0)$ and $Q(x_1, y_1)$, then

$$\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \overrightarrow{QP} = \begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \end{pmatrix} .$$

Inserting this into the equation (5.4), we get

$$\begin{aligned} \operatorname{dist}(P, l) &= \frac{\left|\begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \end{pmatrix}\right|}{\left|\begin{pmatrix} a \\ b \end{pmatrix}\right|} \\ &= \frac{|(x_0 - x_1)a + (y_0 - y_1)b|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_0 + by_0 - ax_1 - by_1|}{\sqrt{a^2 + b^2}} . \end{aligned}$$

The point $Q(x_1, y_1)$ is on the line, i.e. we have

$$ax_1 + by_1 + c = 0 \qquad \Leftrightarrow \qquad c = -ax_1 - by_1 ,$$

wherefore the distance is

$$\operatorname{dist}(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} . \qquad \blacksquare$$

Example 5.14 The distance from the point $P(3, 1)$ to the line

$$l : \quad 5x + 12y - 6 = 0$$

may be found using theorem 5.13. We insert coordinates $x_0 = 3$, $y_0 = 1$, and the constants $a = 5$, $b = 12$ and $c = -6$ into the formula:

$$\operatorname{dist}(P, l) = \frac{|5 \cdot 3 + 12 \cdot 1 - 6|}{\sqrt{5^2 + 12^2}} = \frac{|21|}{\sqrt{169}} = \frac{21}{13} .$$

Example 5.15 The two lines

$$\begin{aligned} l &: \quad 4x - 3y + 2 = 0 \\ m &: \quad 8x - 6y - 5 = 0 \end{aligned}$$

are parallel. This is apparent because their normal vectors

$$\vec{n}_l = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \text{og} \quad \vec{n}_m = \begin{pmatrix} 8 \\ -6 \end{pmatrix} ,$$

are parallel ($\vec{n}_m = 2\vec{n}_l$). So, the two lines are at a fixed distance from each other.

We may also use theorem 5.13 to determine this distance. First, we find a point on one of the lines, and then we determine the distance from this point to the other line (see figure 5.7).

Setting $x = 1$, we get this equation from the line l:

$$4 \cdot 1 - 3y + 2 = 0 \qquad \Longleftrightarrow \qquad y = 2\,.$$

The line l passes through the point $(1, 2)$. We use this point in the formula along with the constants from the equation of m, and we get

$$\operatorname{dist}\big((1;2), m\big) = \frac{|8 \cdot 1 - 6 \cdot 2 - 5|}{\sqrt{8^2 + (-6)^2}} = \frac{|-9|}{\sqrt{100}} = \frac{9}{10}\,.$$

So, the distance between the two lines is $\frac{9}{10}$.

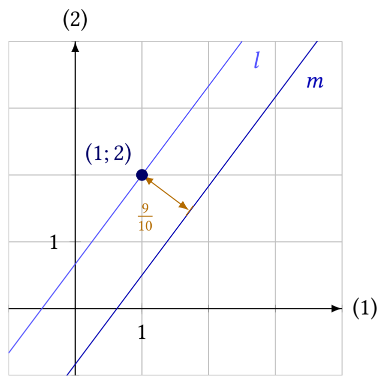


Figure 5.7: The distance from l to m equals the distance from a point on l to m.

5.4 Exercises

Exercise 5.1
Determine the distance between

a) $A(8, 3)$ and $B(2, 9)$,

b) $P(-4, 0)$ and $Q(5, -7)$.

Exercise 5.2
Write a parametrisation of the line with direction vector $\vec{r} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, which passes through the point $P(9; 2)$.

Exercise 5.3
Write a parametrisation for the line passing through the points

a) $A(-5, 3)$ and $B(1, 9)$,

b) $P(4, 7)$ and $Q(5, -3)$.

Exercise 5.4
Write a parametrisation for the line passing through $P(2, 7)$ with normal vector $\vec{n} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Exercise 5.5
Two lines l and m are given by the parametrisations

$$l: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}$$

and

$$m: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \end{pmatrix} + s \cdot \begin{pmatrix} 7 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}\,.$$

a) Determine the intersection point between the two lines.

Exercise 5.6
The line l is given by the parametrisation

$$l: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}\,.$$

a) Determine a parametrisation of the line m passing through $P(11, 5)$, perpendicular to l.

b) Determine the intersection point between l and m.

Exercise 5.7
Determine an equation of the line passing through $P(8,2)$ with normal vector $\vec{n} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.

Exercise 5.8
The line l is given by the parametrisation

$$l: \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad t \in \mathbb{R}.$$

a) Determine an equation for the line m passing through $P(7,0)$ perpendicular to l.
b) Determine the intersection point between l and m.

Exercise 5.9
Write the equations of the following lines in the form $y = \alpha x + \beta$ if it is possible.

a) $3x - 6y + 12 = 0$
b) $-2x + y - 3 = 0$
c) $4x - 20 = 0$
d) $5x - 3y + 9 = 0$.

Exercise 5.10
The line m is perpendicular to the line l given by the equation

$$l: \quad y = 4x - 5.$$

a) Determine the slope of m.

The line m passes through $(0,15)$.

b) Determine an equation of m.
c) Determine the intersection point of m and l.

Exercise 5.11
Determine the distance from the point P to the line l when

a) $P(4,5)$ and $l: \; 3x - 4y + 2 = 0$
b) $P(-1,0)$ and $l: \; 13x + 5y - 24 = 0$
c) $P(9,2)$ and $l: \; 5x - 2y = 6$

6 Circles

A circle may be described as a collection of all the points in its circumference. These points all have the same distance (radius) to a common point (the centre). We may use this to write an equation for a circle:

Theorem 6.1

The equation for the circle centred at $C(x_0, y_0)$ with radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \ .$$

Proof
The circle consists of every point $P(x, y)$ whose distance to the centre is r, i.e. $|CP| = r$. But according to theorem 5.1

$$|CP| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \ ,$$

so,

$$\begin{aligned} \sqrt{(x - x_0)^2 + (y - y_0)^2} &= r \qquad \Longleftrightarrow \\ (x - x_0)^2 + (y - y_0)^2 &= r^2 \ . \end{aligned}$$

■

Example 6.2 The equation of the circle with radius $r = 5$ centred at $C(4, -1)$ is

$$\begin{aligned} (x - 4)^2 + (y - (-1))^2 &= 5^2 \qquad \Longleftrightarrow \\ (x - 4)^2 + (y + 1)^2 &= 25 \ . \end{aligned}$$

We may simplify this equation further by multiplying the parentheses. We then get

$$\begin{aligned} x^2 + 4^2 - 2 \cdot x \cdot 4 + y^2 + 1^2 + 2 \cdot y \cdot 1 &= 25 \qquad \Longleftrightarrow \\ x^2 + 16 - 8x + y^2 + 1 + 2y &= 25 \qquad \Longleftrightarrow \\ x^2 + y^2 - 8x + 2y - 8 &= 0 \ . \end{aligned}$$

Example 6.3 The equation

$$x^2 + y^2 - 6x + 14y + 42 = 0 \ ,$$

is the equation of a circle. If we want to find the centre and radius of the circle, we need to rewrite the equation, so that it has the same form as the equation in theorem 6.1.

To do this, we must rewrite the terms x^2 and $-6x$ into the square of a sum or a difference. The same goes for the terms y^2 and $14y$. If x^2 and $-6x$ are the results of squaring a sum or a difference, then $-6x$ is double the product of two terms, we therefore calculate

$$(x-3)^2 = x^2 + 3^2 - 2 \cdot x \cdot 3 = x^2 + 9 - 6x \, ,$$

Here, we see that $x^2 - 6x$ may be written as $(x-3)^2 - 9$. In the same way, we may show that $y^2 + 14y = (y+7)^2 - 49$. Therefore, we may write the equation of the circle as

$$\begin{aligned} x^2 + y^2 - 6x + 14y + 42 = 0 \quad &\Longleftrightarrow \\ x^2 - 6x + y^2 + 14y + 42 = 0 \quad &\Longleftrightarrow \\ (x-3)^2 - 9 + (y+7)^2 - 49 + 42 = 0 \quad &\Longleftrightarrow \\ (x-3)^2 + (y+7)^2 = 16 \quad &\Longleftrightarrow \\ (x-3)^2 + (y-(-7))^2 = 4^2 \, . & \end{aligned}$$

Now that the equation has this form, we may find directly that the radius is $r = 4$, and the centre is at $C(3, -7)$.

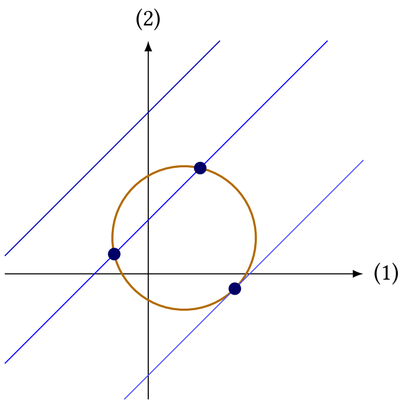


Figure 6.1: A line may intersect a circle at 0, 1 or 2 points.

6.1 Intersections Between Circles and Lines

A circle and a line may intersect at 0, 1 or 2 points, see figure 6.1. If the circle and the line have exactly one common point, the line is tangent to the circle. We may determine the amount of intersection points between a line and a circle by calculating the distance from the centre to the line. If this distance is equal to the radius, the line is a tangent. If the distance is less than the radius, there are two intersection points, and if the distance is greater than the radius, the line and the circle do not intersect.

Example 6.4 Here, we determine whether the line with equation

$$3x - 2y - 7 = 0$$

and the circle with equation

$$(x+1)^2 + (y-2)^2 = 9$$

intersect.

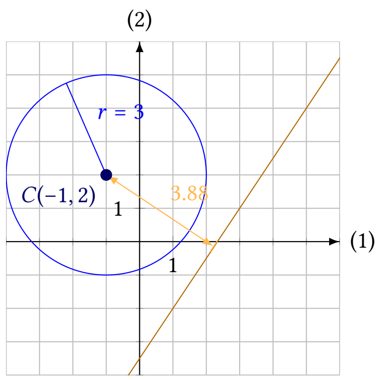


Figure 6.2: The distance from the centre to the line is greater than the radius.

The centre of the circle is at $C(-1, 2)$, and its radius is $r = \sqrt{9} = 3$. The distance from the centre to l is

$$\operatorname{dist}(C, l) = \frac{|3 \cdot (-1) - 2 \cdot 2 - 7|}{\sqrt{3^2 + (-2)^2}} = \frac{|-14|}{\sqrt{13}} = \frac{14}{\sqrt{13}} \approx 3.88 \, .$$

The distance from the centre to the line is greater than the radius, so, the line and the circle do not intersect (see figure 6.2).

Example 6.5 The line with equation

$$x - 3y + 2 = 0$$

and the circle with equation

$$(x-1)^2+(y-4)^2=25$$

intersect at two points. This is true because the distance from the centre of the circle $C(1,4)$ to the line is

$$\operatorname{dist}(C,l)=\frac{|1\cdot 1-3\cdot 4+2|}{\sqrt{1^2+(-3)^2}}=\frac{|-9|}{\sqrt{1+9}}=\frac{9}{\sqrt{10}}\approx 2.85\ ,$$

which is less than the radius of the circle $r=\sqrt{25}=5$ (see figure 6.3).

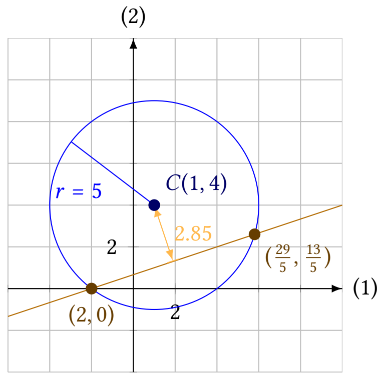


Figure 6.3: The distance from the centre to the line is less than the radius.

Example 6.6 The line and the circle in example 6.5 intersect at two points. Here, we find the coordinates of these intersection points (see figure 6.3).

First, we rewrite the equation of the line

$$x-3y+2=0 \qquad \Longleftrightarrow \qquad x=3y-2\ .$$

Next, we insert this value of x into the equation of the circle and get

$$(3y-2-1)^2+(y-4)^2=25\ .$$

This is a quadratic equation. Using a CAS, we solve the equation and get

$$y=0 \quad \vee \quad y=\frac{13}{5}\ .$$

The two intersection points have these y-coordinates. The x-coordinates may then be determined using the equation of the line:

$$\begin{aligned} y=0:&\quad x=3\cdot 0-2=-2\\ y=\frac{13}{5}:&\quad x=3\cdot\frac{13}{5}-2=\frac{39}{5}-\frac{10}{5}=\frac{29}{5}\ . \end{aligned}$$

So, the line intersects the circle at the two points $(-2,0)$ and $(\frac{29}{5},\frac{13}{5})$.

If the line is instead described by a parametrisation, we cannot directly calculate the distance from the line to the centre of a given circle. If we want to find this distance, we then need to find an equation of the line (see example 5.9).

But if we only want the intersection points between the line and the circle, this is not necessary. Here, we instead use the coordinate functions of the parametrisation.

Example 6.7 If a line is given by the parametrisation

$$\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} 3\\ 5\end{pmatrix}+t\begin{pmatrix} 2\\ -1\end{pmatrix}\ ,\qquad t\in\mathbb{R}\ .$$

and a circle is given by the equation

$$(x-2)^2+(y-1)^2=16\ ,$$

we may find the intersection points by inserting the coordinate functions of the line

$$x = 3 + 2t \quad \text{and} \quad y = 5 - t$$

into the equation of the circle. We therefore insert the two expressions for x and y into the equation of the circle:

$$\begin{aligned}
(\overbrace{3 + 2t}^{x} - 2)^2 + (\overbrace{5 - t}^{y} - 1)^2 = 16 \quad &\Longleftrightarrow \\
(1 + 2t)^2 + (4 - t)^2 = 16 \quad &\Longleftrightarrow \\
1 + 4t^2 + 4t + 16 + t^2 - 8t = 16 \quad &\Longleftrightarrow \\
5t^2 - 4t + 17 = 16 \quad &\Longleftrightarrow \\
5t^2 - 4t + 1 = 0 \ .
\end{aligned}$$

Solving this via CAS, we find the solutions

$$t = -\tfrac{1}{5} \quad \vee \quad t = 1 \ .$$

Now we know there are two intersection points because the two solutions are the values of the parameter t, which correspond to the intersection points. To determine the points, we insert these values of t into the parametrisation of the line:

$$\begin{aligned}
t = -\tfrac{1}{5} : \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \left(-\tfrac{1}{5}\right) \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ \frac{26}{5} \end{pmatrix} \\
t = 1 : \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \ .
\end{aligned}$$

So, the line intersects the circle at the points $\left(\frac{8}{5}, \frac{26}{5}\right)$ and $(5, 4)$.

6.2 Circle Tangents

A tangent of a circle is a line which has exactly one point in common with the circle. We call this point the *point of tangency*. Moreover, the tangent of a circle will always be perpendicular to the line through the centre and the point of tangency.

As previously mentioned, we may determine whether a given line is a tangent by calculating the distance from the line to the centre of the circle. The distance from the centre to a tangent is equal to the radius. Alternatively, we may determine whether a line given by a parametrisation is a tangent by determining the number of intersection points (see example 6.7).

The following examples show how to *construct* tangents when we know either a point or a slope.

Example 6.8 The point $P(5, -7)$ lies on the circle given by the equation

$$(x - 2)^2 + (y + 3)^2 = 25 \ .$$

Here, we determine an equation of the tangent to the circle, which has this point as its point of tangency.

The centre of the circle is $C(2,-3)$. The tangent is perpendicular to the line segment CP, i.e. the vector $\overrightarrow{CP}$ is a normal vector of the tangent:

$$\overrightarrow{CP} = \begin{pmatrix} 5-2 \\ -7-(-3) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} .$$

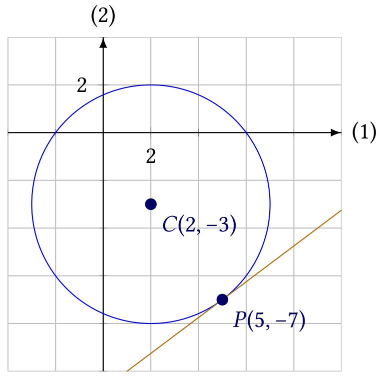


Figure 6.4: The tangent through $P(5,-7)$ of the circle.

Now we know a point $P(5,-7)$, which the line passes through, and we now a normal vector, $\overrightarrow{CP}$. So, using theorem 5.7, we get this equation:

$$\begin{aligned} 3(x-5)+(-4)\cdot(y-(-7)) &= 0 && \iff \\ 3x-4y-43 &= 0 . \end{aligned}$$

The circle and the tangent are illustrated in figure 6.4.

Example 6.9 Here, we determine the points of tangency for those tangents of

$$(x-2)^2+(y-1)^2 = 25 ,$$

which are parallel to the line l with equation

$$l : \quad 4x-3y+30 = 0 .$$

The circle must have two tangent parallel to the given line. A line through the points of tangency must pass through the centre and be perpendicular to the line l. Since this line is perpendicular to l, we may find a parametrisation of the line by using the normal vector of l as a direction vector. I.e.

$$\vec{r} = \vec{n}_l = \begin{pmatrix} 4 \\ -3 \end{pmatrix} .$$

We know that the line through the points of tangency pass through the centre $C(2,1)$. This point, along with the direction vector, provides the parametrisation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} , \qquad t \in \mathbb{R} .$$

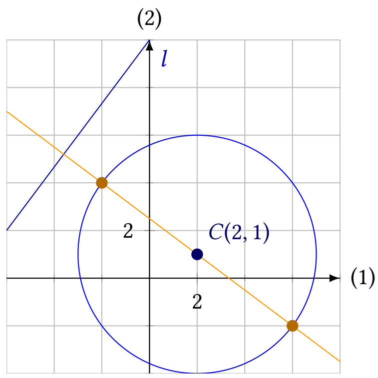


Figure 6.5: The points of tangency for those tangents of the circle parallel to the line l.

The intersection points between this line and the circle are the points of tangency we are looking for (see figure 6.5). These intersection points may be determined using the same method as in example 6.7. The coordinate functions of the line

$$x = 2+4t \qquad \text{and} \qquad y = 1-3t ,$$

are inserted into the equation of the circle:

$$\begin{aligned} (2+4t-2)^2+(1-3t-1)^2 &= 25 && \iff \\ (4t)^2+(-3t)^2 &= 25 && \iff \\ 16t^2+9t^2 &= 25 && \iff \\ 25t^2 &= 25 && \iff \\ t^2 &= 1 && \iff \end{aligned}$$

$$t = -1 \quad \vee \quad t = 1\ .$$

We now insert these two values of the parameter t into the parametrisation, and get

$$\begin{aligned} t = -1 : \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ t = 1 : \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}\ . \end{aligned}$$

So, the points of tangency of the two tangents parallel to l are $(-2, 4)$ and $(6, -2)$. If we want to determine the equations of the tangents, we may do so by using the method in example 6.8.

6.3 Parametrisation of a Circle

Concluding this chapter, we show how to describe circles by parametrisations.

The vector with length r and directional angle t has coordinates $r \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$. This vector is the position vector of a point on the circle with radius r centred at $(0, 0)$. If the angle t runs through the interval from $-180°$ to $180°$, we get every point on the circumference of the circle. I.e. this circle may be described by the parametrisation

$$\begin{pmatrix} x \\ y \end{pmatrix} = r \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \quad -180° < t \leq 180°\ .$$

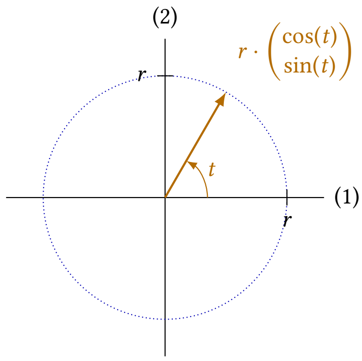


Figure 6.6: Parametrisation of the circle with radius r centred at $(0, 0)$.

If we want to derive a parametrisation of the circle centred instead at (x_0, y_0), we only need to add the position vector of this point to the parametrisation above, and we get the following theorem:

Theorem 6.10

The circle with radius r centred at $C(x_0, y_0)$ may be described by the parametrisation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\cos(t) \\ r\sin(t) \end{pmatrix}, \quad -180° < t \leq 180°\ .$$

This is not the only possible parametrisation of the circle. Actually, any parametrisation of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \end{pmatrix}$$

describes the circle. A parametrisation such as this is used to describe a point moving with constant speed in a circle. The magnitude of the constant ω is linked to the speed of the point on the circle.

6.4 Exercises

Exercise 6.1
Write an equation of the circle centred at C with radius r when

a) $C(1,4)$ and $r = 2$

b) $C(0,6)$ and $r = 7$

c) $C(-3,7)$ and $r = 1$

Exercise 6.2
Determine the centres and the radii of the circles given by the following equations:

a) $x^2 + y^2 - 6x + 4y = 23$

b) $x^2 + y^2 + 2x = 24$

c) $x^2 + y^2 + 8x + 4y = 61$

Exercise 6.3
Determine the number of intersection points between the line l given by

$$l : \quad 4x - 7y + 8 = 0$$

and the circle given by

$$(x-1)^2 + (y-5)^2 = 16 .$$

Exercise 6.4
Determine the intersection points between the line l given by the equation

$$l : \quad x + y = 6$$

and the circle given by the equation

$$(x-3)^2 + (x-2)^2 = 13 .$$

Exercise 6.5
Determine the intersection points between the line l given by the parametrisation

$$l : \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ -9 \end{pmatrix} , \quad t \in \mathbb{R}$$

and the circle given by the equation

$$(x+1)^2 + (y-4)^2 = 25 .$$

Exercise 6.6
The circle with equation

$$(x-2)^2 + (y-3)^2 = 25$$

passes through the three points $A(-2,6)$, $B(2,8)$ og $C(7,3)$.

a) Determine an equation of the tangent to the circle at each of these points.

Exercise 6.7
The circle with equation

$$(x-5)^2 + (y+1)^2 = 169 ,$$

has two tangents parallel to the line with equation

$$-12x + 5y + 10 = 0 .$$

a) Determine the point of tangency for each of these tangents.

b) Determine an equation for each tangent.